International Journal of Fluid Mechanics Research, Vol. 28, Nos.1-2, 2001

Wave Disturbances of Stratified Fluid due to Vertical Jet[†]

I. T. Selezov and S. G. Shpakova
Institute of Hydromechanics
of National Academy of sciences of Ukraine, Kyiv

P. Huq
The Graduate College of Marine Stidies Robinson Hall
University of Delaware, Newark, USA

The statement and solution of a new problem are presented for fluid wave motions at the injection of vertical axisymmetric jet into a stable stratified fluid, which density linearly increases with the depth. Flow in the jet is assumed to be potential, the motion of stratified fluid is described by Boussinesq's approximation. The dispersion equation is derived and analysed. The conditions of wave existence are found and the analysis of phase and group velocities and wave modes is presented. It is shown that in outer medium the wave disturbances propagate along the jet as it was observed in experiments.

* * *

Introduction

The problem of jet flows is of great theoretical and applied importance and it has been considered in numerous investigations.

There are many works devoted to several aspects of the jet analysis such as supersonic jets in gas, turbulent jets, jet stability, experimental investigations. At the same time, a considerable number of works deal with plane jets. Analysis of these works is beyond the scope of this paper.

We consider some works relating only to circular jets in the cases of laminar regime or potential flow corresponding to the initial stage of smooth directed jet intrusion before the jet is able to be extended and evolved. Also, some works for jets in the presence of crossflow are considered.

Some self-similar solutions are presented [1 – 6] in most of theoretical works. For example, in [1] the self-similar solution has been obtained for the case of large radial non-axisymmetric oscillations of fluid column - jet (plane problem) so that the region boundary is the unknown function to be determined. The fluid is assumed to be inviscid and incompressible. In general there is a few of known semi-similar solutions but there are many works analysing the jet behaviour on the basis

[†] Translated from *Prykladna Gidromekhanika*, 1999, 1(73), No. 2, pp. 38–44.

ISSN 1064-2285

of these solutions.

Some analytical solutions for circular jets have been obtained in [6 – 12]. The theory of a thin jet with surface tension is developed in [9]. The potential motion of axisymmetric jet externally bounded by the free surface $r = R(x, t)$ is considered, at that the external pressure is assumed to be constant and the surface tension is taken into account. The surface and the flow are assumed to be changed smoothly along the axis that it is characterized by the small parameter ε -- (ratio of the radial and axial scales). The potential φ and the radius R are expanded in power series on ε^2 and as a result the second order partial differential equation is derived representing the theory of thin jets in terms of leading order terms. In [13, 14] the development of capillary waves in axisymmetric liquid jet is investigated in nonlinear statement. The paper [13] developes a robust approximation for the injection of vertical jet into atmosphere in the case when the influence of the external medium on a jet stability is not taken into account. Also, in doing so the fluid is assumed to be inviscid and incompressible and the flow potential. In this case the initial axisymmetric disturbance of the surface is assumed to be given in the form of standing sinusoidal wave. The solutions for the jet surface deviation η and the potential φ are searched in the form of expansions in the small amplitude parameter η_0 of the given standing wave. The paper [7] presents the statement and numerical solution of the problem for laminar axisymmetric jet. The vertical stationary jet intruding with a mean axial velocity into another fluid is considered. Both of the fluids are assumed to be incompressible, immiscible, Newtonian. The velocity profiles are constructed and analysed. In [10, 11] the numerical and experimental analysis of the laminar circular jet with taking into account the viscosity and surface tension is carried out. In [11] the problem of the vertical circular jet injection into the resting fluid is investigated. The case is analysed when the standing waves on the jet surface are given at initial state. Both of the fluids are assumed to be viscous (Newtonian) and incompressible. The initial-boundary value problem is solved using the exact solution in closed form, having obtained for inviscid fluids in [15], and the analytical solution for viscous fluids obtained in [16].

The developing structure of accelerating jet at a crossflow investigated in [17]. In [8] the jet behaviour in crossflow is investigated on the basis of asymptotic method. The fluid is assumed to be incompressible and inviscid, the flow potential. The velocity components of the external flow, normal to jet axis, are taken to be small relative to the jet velocity that allows to introduce a small parameter. Asymptotic solutions in near and far zones are developed.

Experimental investigations of [18] reveal that near the inlet, at the initial part of injection, the jet does not deviate from the vertical position and keeps a stable form even in the presence of a crossflow, i. e. the initial part may be considered as a zone of stable potential flow. However, waves may propagate along the interface.

Earlier it was noted in [19] that streamlines at the potential zone are strongly parallel to the jet direction, and one can suppose that the cross-flow does not distort the jet, that is the stability at the initial part takes place. In what follows the jet is expanded and deviates primarily in the laminar regime, then in the turbulent ones, and, lastly, disintegrates and drifts down the flow. At the same time in [18] wave disturbances were observed in a jet vicinity.

In this paper the initial stage of the injection is investigated so that the jet is assumed to be axisymmetric in agreement with considerations above presented.

1. Statement of Problem

The axisymmetric problem for vertical jet of the diameter D injected into a steady density stratified fluid is considered. The problem is solved in linearized statement. It is convenient to introduce the cylindrical coordinate system r, θ, z, so that the z-axis coincides with the jet axis and it is directed downward. The region occupied by the jet is

$$\Omega^j = \{r, \theta, z | r \in [0, D/2], \theta \in [0, 2\pi], z \in (-\infty, \infty)\}.$$

The external medium occupies the region

$$\Omega = \{r, \theta, z | r \in [D/2, \infty), \theta \in [0, 2\pi], z \in (-\infty, \infty)\}.$$

The fluid in the jet is considered to be inviscid, incompressible, the undisturbed velocity U_0 is assumed to be uniformly distributed over the cross-section of the jet. This allows to introduce the total potential Φ by the formula

$$\vec{v} = \vec{\nabla}\Phi^j$$

and by virtue of considering linear problem to present the total field as a superposition of undisturbed and disturbed components [4]

$$\Phi^j(r, z, t) = U_0 z + \varphi^j(r, z, t),$$

$$\frac{\partial \Phi^j}{\partial z} = U_0 + v_z^j,$$

$$\frac{\partial \Phi^j}{\partial r} = \frac{\partial \varphi^j}{\partial r} = v_r^j.$$

The disturbed motion is described by the potential φ^j satisfying the equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \varphi^j}{\partial r}\right) + \frac{\partial^2 \varphi^j}{\partial z^2} = 0 \quad \text{in} \quad \Omega^j. \tag{1}$$

As the governing equations for the motion of stratified medium, we take the linear equations of hydrodynamics of incompressible, inviscid fluid in Boussinesq approximation [20]. These equations in the cylindrical coordinate system in the region Ω are written in the form

$$\frac{\partial v_r}{\partial t} + \frac{\partial p}{\partial r} = 0, \tag{2}$$

$$\frac{\partial w}{\partial t} + \frac{\partial p}{\partial z} + \rho = 0, \tag{3}$$

$$\frac{\partial \rho}{\partial t} = N^2 w, \tag{4}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) = -\frac{\partial w}{\partial z}, \tag{5}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) = \frac{\partial^2 w}{\partial t \partial z}, \tag{6}$$

where ρ and p are the density and the pressure, w and v_r are the velocity components of stratified fluid motion along the axes z and r, respectively, N is the Brunt–Vaisala buoyancy frequency. After simple transformations the system of equations (2)–(6) can be reduced to the following resolving equation

$$\frac{\partial^2}{\partial t^2}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{\partial^2 w}{\partial z^2}\right] + N^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)\right] = 0. \tag{7}$$

The matching conditions at the fluid interface $r = 1/2$ are of the form [21]

$$\left(\frac{\partial^2\varphi^j}{\partial t\partial r} - U_0\frac{\partial v_r}{\partial z} - \frac{\partial v_r}{\partial t}\right)_{r=\frac{1}{2}} = 0, \tag{8}$$

$$\rho_0^j\left(\frac{\partial\varphi^j}{\partial t} + U_0\frac{\partial\varphi^j}{\partial z}\right)_{r=\frac{1}{2}} = -p|_{r=\frac{1}{2}}, \tag{9}$$

where ρ_0^j is the fluid density in the jet. Moreover, we define the regularity conditions on the axis and the boundedness condition of the functions at infinity

$$\frac{\partial\varphi^j}{\partial r} = 0 \text{ at } r = 0, \quad v_r, w, p < \infty \text{ at } r \to \infty. \tag{10}$$

It should be noted that all the values in Eqs. (1)–(10) and further are dimensionless introduced by the formulae (asterisks are omitted)

$$\begin{gathered}(r^*, z^*) = \frac{1}{D}(r, z), \; t^* = \sqrt{\frac{g}{D}}, \; p^* = \frac{1}{\rho_{00}gD}p, \\ \omega^* = \sqrt{\frac{D}{g}}\omega, \; \rho^* = \frac{\rho}{\rho_{00}}, \; \rho_0^{*j} = \frac{\rho_0^j}{\rho_{00}}, \\ (v_r^*, w, U_0^*) = \frac{1}{\sqrt{gD}}(v_r, w, U_0), \; k^* = Dk, \\ N^{*2}(z^*) = \frac{D}{g}N^2(z), \; \varphi^* = \frac{1}{D\sqrt{gD}}\varphi,\end{gathered} \tag{11}$$

where ρ_{00} is the undisturbed density of stratified medium.

2. Investigation of the Running Wave Propagation

We consider the solutions of Eqs. (1), (7) in the class of traveling waves along the axis z

$$\{\varphi^j, w\} = \{\hat{\varphi}^j(r), \hat{w}(r)\}\, exp\left[i(kz - \omega t)\right], \tag{12}$$

where k and ω are the wave number and the angular frequency.

Substituting Eq. (12) to Eqs. (1) and (7)–(9), respectively, leads to the following system

$$\frac{d^2\hat{\varphi}^j}{dr^2} + \frac{1}{r}\frac{d\hat{\varphi}^j}{dr} - k^2\hat{\varphi}^j = 0 \quad \text{in} \quad \Omega^j, \tag{13}$$

$$\frac{d^2\hat{w}}{dr^2} + \frac{1}{r}\frac{d\hat{w}}{dr} + \frac{k^2}{N^2/\omega^2 - 1}\hat{w} = 0 \quad \text{in} \quad \Omega, \tag{14}$$

$$\left.\frac{d\hat{\varphi}^j}{dr}\right|_{r=\frac{1}{2}} + \frac{k}{\omega}\left(U_0 - \frac{\omega}{k}\right)\hat{v}_r|_{r=\frac{1}{2}} = 0, \tag{15}$$

$$ik\rho_0^j\left(U_0 - \frac{\omega}{k}\right)\hat{\varphi}^j|_{r=\frac{1}{2}} = -\hat{p}|_{r=\frac{1}{2}}. \tag{16}$$

Taking into account the regularity condition on the axis (10) the solution of equation (13) is written in the form [22]

$$\hat{\varphi}^j(r) = C_1 I_0(kr). \tag{17}$$

Crossing to a new variable

$$r_1 = k_1 r, \quad k_1 = \frac{k}{\sqrt{N^2/\omega^2 - 1}}, \tag{18}$$

transforms the equation (14) to the following form

$$\frac{d^2\hat{w}}{dr_1^2} + \frac{1}{r_1}\frac{d\hat{w}}{dr_1} + \hat{w} = 0. \tag{19}$$

The solutions of the equation (19) may be written both in terms of the Bessel functions of the first kind and in terms of the Bessel functions of the third kind, that is in the Hankel functions [22]. The former one describes standing transverse waves, the latter one describes traveling transverse waves.

Let us consider the solution of (19) in the Hankel functions. Taking into account the conditions (10) the solution is written in the form

$$\hat{w}(r) = C_3 H_0^{(1)}(k_1 r). \tag{20}$$

Substituting Eq. (20) into Eqs. (2)–(4) allows to determine the other searched functions

$$\hat{\rho}(r) = i\frac{N^2}{\omega} C_3 H_0^{(1)}(k_1 r), \tag{21}$$

$$\hat{v}_r(r) = -i\frac{k}{k_1} C_3 H_1^{(1)}(k_1 r), \tag{22}$$

$$\hat{p}(r) = -\frac{\omega k}{k_1^2} C_3 H_0^{(1)}(k_1 r). \tag{23}$$

It should be noted that by writing the solution in the Hankel functions we exclude thus from the consideration imaginary values of the wave number k [24]. The radicand in the coefficient k_1 (18) is always positive since internal waves can exist only with the frequency $\omega < N$ [20,25].

Substituting the solutions (17), (22), (23) into the matching conditions (15), (16) yields the dispersion equation of the form

$$\frac{I_1(k/2)}{I_0(k/2)} + \frac{\rho_0^j (U_0/c - 1)^2}{\sqrt{N^2/\omega^2 - 1}} \frac{H_1^{(1)}(k_1/2)}{H_0^{(1)}(k_1/2)} = 0. \tag{24}$$

Let us investigate this equation. At any k the first term in (24) is always positive:

$$I_0(k/2)/I_1(k/2) > 0.$$

Due to $N^2/\omega^2 - 1 > 0$, the coefficient in front of the ratio of the Hankel functions is also always positive. So, the equation (24) can be written in the form

$$1 + A\frac{H_1^{(1)}(k_1/2)}{H_0^{(1)}(k_1/2)} = 0, \tag{25}$$

where A is a positive coefficient. Representing the Hankel functions by the Bessel functions according to the formula

$$H_m^{(1)}(\gamma) = J_m(\gamma) + iY_m(\gamma), \qquad (m = 0, 1),$$

allows to reduce the equation (25) to the system of two equations

$$J_0(k_1/2)\left[1 + A\frac{J_1(k_1/2)}{J_0(k_1/2)}\right] = 0, \tag{26}$$

$$Y_0(k_1/2)\left[1 + A\frac{Y_1(k_1/2)}{Y_0(k_1/2)}\right] = 0. \tag{27}$$

There are four cases of solvability of this system:

Case 1.

$$J_0(k_1/2) = 0, \qquad Y_0(k_1/2) = 0.$$

It is known from the theory of Bessel functions that positive roots of two linearly independent real cylindrical functions of the same order are alternate [22]. It follows from above that there is no such $k_1 > 0$, when $J_0(k_1/2) = 0$ and $Y_0(k_1/2) = 0$ simultaneously.

Case 2.

$$1 + A\frac{J_1(k_1/2)}{J_0(k_1/2)} = 0, \qquad 1 + A\frac{Y_1(k_1/2)}{Y_0(k_1/2)} = 0, \tag{28}$$

Let us consider the Wronskian [22]

$$J_1(\gamma)Y_0(\gamma) - J_0(\gamma)Y_1(\gamma) = \frac{2}{\pi\gamma}, \tag{29}$$

which is transformed to the form

$$J_0(\gamma)Y_0(\gamma)\left[\frac{J_1(\gamma)}{J_0(\gamma)} - \frac{Y_1(\gamma)}{Y_0(\gamma)}\right] = \frac{2}{\pi\gamma}. \tag{30}$$

As far as the Wronskian (29) is not equal to zero, then always

$$\frac{J_1(\gamma)}{J_0(\gamma)} \neq \frac{Y_1(\gamma)}{Y_0(\gamma)}.$$

Thus, there are no such real values of $k_1 > 0$ which turn into zero both of the equations of (28) simultaneously.

Case 3.

$$J_0(k_1/2) = 0, \qquad 1 + A\frac{Y_1(k_1/2)}{Y_0(k_1/2)} = 0,$$

It is seen from the graphs and tables of the Bessel functions $J_m(\gamma)$,$Y_m(\gamma)$ $(m = 0, 1)$ [22,26] that the roots of these Bessel functions are placed by the following manner: if $k_1/2$ is the root of the function $J_0(\gamma)$ then the values of the functions $Y_0(\gamma)$, $Y_1(\gamma)$ in this point are either of the same sign or $k_1/2$ is also the root of the function $Y_1(\gamma)$. Hence the second expression will be different from zero. So, in this casethe system also does not have any solution.

Case 4.

$$Y_0(k_1/2) = 0, \qquad 1 + A\frac{J_1(k_1/2)}{J_0(k_1/2)} = 0.$$

In this case we observe a picture analogous to considered one in the previous variant. If $k_1/2$ is the root of the function $Y_0(\gamma)$ then the values of the functions $J_0(\gamma)$, $J_1(\gamma)$ in this point are either of the same sign or $k_1/2$ is also the root of the function $J_1(\gamma)$ and the second expression will be different from zero.

On the basis of stated above the conclusion can be made that the system of equations (26), (27) is unsolvable and, a consequence, the dispersion equation (24) has no solutions in the region of real values $k_1 > 0$.

Let us now write the solution of equation (19) in terms of the Bessel functions of the first kind. Having carried out the same algebra yields the dispersion equation of the form

$$\frac{I_1(k/2)}{I_0(k/2)} + \frac{\rho_0^j (U_0/c-1)^2}{\sqrt{N^2/\omega^2-1}} \frac{J_1(k_1/2)}{J_0(k_1/2)} = 0. \tag{31}$$

Crossing in the equation (31) from the angular frequency ω and the wave number k to the phase velocity c and the wave length λ by the formulae $k = 2\pi/\lambda, \omega = c2\pi/\lambda$ leads to the following equation

$$\sqrt{(N\lambda/2\pi c)^2-1}\frac{I_1(\pi/\lambda)}{I_0(\pi/\lambda)}.J_0\left(\frac{\pi/\lambda}{\sqrt{(N\lambda/2\pi c)^2-1}}\right)+$$

$$+\rho^j(\frac{U_0}{c}-1)^2 J_1\left(\frac{\pi/\lambda}{\sqrt{(N\lambda/2\pi c)^2-1}}\right) = 0 \tag{32}$$

3. Analysis of Wave Dispersion

To simplify the following analysis of the dispersion equation (32) we introduce the replacement

$$x = \frac{\pi/\lambda}{\sqrt{(N\lambda/2\pi c)^2-1}} \tag{33}$$

and transform the equation (32) to the form

$$J_1(x) + \varepsilon(x)J_0(x) = 0, \tag{34}$$

where

$$\varepsilon(x) = \frac{N^2\left(\frac{\lambda}{\pi}\right)^3 x}{\rho_0^j\left[2U_0\sqrt{1+\left(\frac{\lambda}{\pi}x\right)^2} - N\left(\frac{\lambda}{\pi}\right)^2 x\right]^2}\frac{I_1\left(\frac{\pi}{\lambda}\right)}{I_0\left(\frac{\pi}{\lambda}\right)}. \tag{35}$$

The coefficient $\varepsilon(x)$ in the dispersion equation (34) is always positive. So, the left-hand side of the equation (34) can change its sign only at different signs of the values $J_0(x), J_1(x)$. It is seen from graphs of the functions $J_0(x), J_1(x)$ [27] that the real root of (34) is between the roots of the Bessel functions $J_0(x), J_1(x)$. Then from the equation (32) we obtain the estimate for the bounds of phase velocity

$$\frac{N\left(\frac{\lambda}{\pi}\right)^2}{2\sqrt{\left(\frac{\lambda}{\pi}\right)^2+1/(\chi_{0i})^2}} < c < \frac{N\left(\frac{\lambda}{\pi}\right)^2}{2\sqrt{\left(\frac{\lambda}{\pi}\right)^2+1/(\chi_{1i})^2}} \qquad (i=1,2,\ldots), \tag{36}$$

where χ_{0i}, χ_{1i} $(i = 1, 2, \ldots)$ are the roots of the Bessel functions $J_0(x), J_1(x)$. It is seen from formula (36) that the phase velocity depends on the wave length, the buoyancy frequency N and the value of the roots of the Bessel functions.

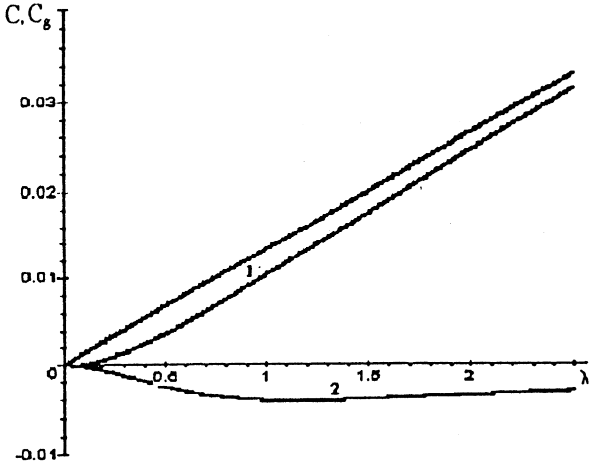

Fig. 1. Pahse velocity C (curve *1*) and group velocity C_g (curve *2*) versus wave length λ.

Let us estimate the order of the coefficient $\varepsilon(x)$ according to data presented in [18]: the buoyancy frequency $N \cong 0.1$, the jet density $\rho_0^j \cong 1$, and the flow velocity $U_0 \cong 6$. In this case the ratio

$$\frac{I_1(\pi/\lambda)}{I_0(\pi/\lambda)} \leq 1 \text{ and } \frac{I_1(\pi/\lambda)}{I_0(\pi/\lambda)} = 1 \text{ when } \lambda \to 0,$$

$$\frac{I_1(\pi/\lambda)}{I_0(\pi/\lambda)} \to \frac{1}{2}\pi/\lambda \text{ when } \lambda \to \infty.$$

Therefore

$$\varepsilon(x) \cong \frac{N^2 \left(\frac{\lambda}{\pi}\right)^3 x}{4\rho_0^j U_0^2} \text{ when } \lambda \to 0,$$
$$\varepsilon(x) \cong \frac{1}{2\rho_0^j x \left(\frac{\lambda}{\pi}\right)^2} \text{ when } \lambda \to \infty. \tag{37}$$

The values of the coefficient $\varepsilon(x)$ have been calculated in a large range of wave length λ. As a result it was determined that in the range under consideration the value $\varepsilon(x)$ does not exceed the order of 10^{-5}. This allows us to take the value of the root of the Bessel function $J_1(x)$ as the solution of the equation (28) with an error not exceeding the order of 10^{-5}. Thus with the same degree of accuracy we may determine the phase velocity c

$$c = \frac{N\chi_{1i}(\lambda/\pi)^2}{2\sqrt{1+(\lambda\chi_{1i}/\pi)^2}}. \tag{38}$$

The group velocity is determined by the formula

$$c_g = -\frac{c}{1+(\lambda\chi_{1i}/\pi)^2}. \tag{39}$$

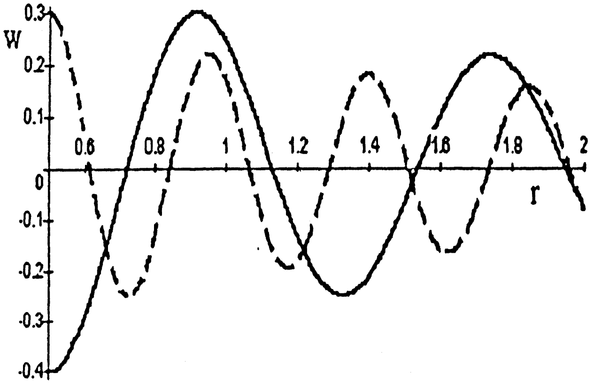

Fig. 2. Behaviour of normal mode in radial direction in the external medium: solid curve – the 1-st mode, dashed curve – the 2-nd mode.

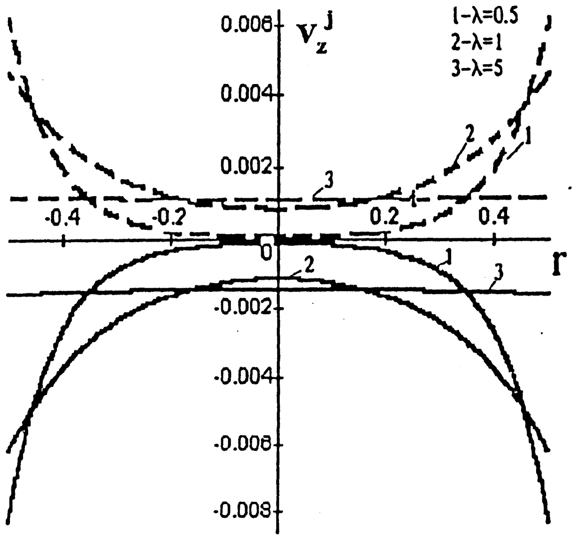

Fig. 3. Change of normal mode in the jet v_z^j in radial direction: solid curves – the 1-st mode, dashed curve – the 2-nd mode.

It follows from the analysis of the dispersion equation that the plane $c(\lambda), \lambda$ (Fig. 1) is separated onto two regions by the curve $c = N\lambda/(2\pi)$. In the upper region $c > N\lambda/(2\pi)$ solutions do not exist, in the lower region the curve $c(\lambda)$ is close to the asymptote, with changing λ from 0 to ∞ the value $c(\lambda)$ deviate first down from the asymptote, then it has a point of inflection and approacheing the line $c = N\lambda/(2\pi)$.

It may be revealed from the analysis of group velocities that for waves propagating in the positive direction the group velocity is negative. Therefrom it follows that the energy transport takes place in the negative direction that is the back wave takes place.

Figs. 2, 3 represent the wave modes. It should be noted that the wave modes in the jet and in

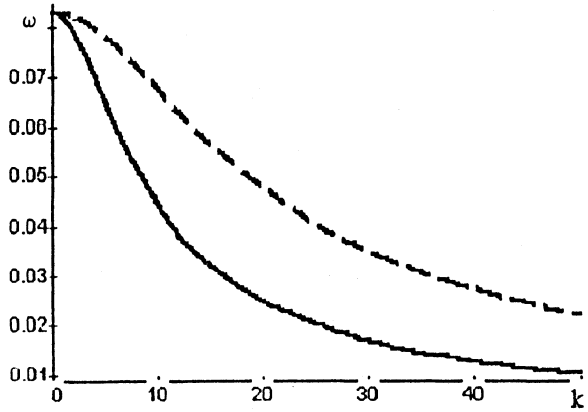

Fig. 4. Wave frequency ω versus wave number k:
solid curve – the 1-st mode, dashed curve – the 2-nd mode.

the external medium are appreciably different. Forms of the wave modes in the jet depend on the wave length while the wave modes in the external medium are independent of λ. The external mode decreases oscillating as $1/\sqrt{r}$ and at the distance $10D$ from the jet surface it decreases fivefold. Moreover, the mode amplitudes in the jet are several order less than in the external medium (10^{-3}). The conclusion could be drawn from here that the transport of wave energy in the jet is sufficiently less than in the external region adjoining the jet.

Fig. 4 presents the dependence of the angular frequency ω on the wave number k. It follows from this Figure that the phase and group velocities steadily decreasing with k. The analogous picture of the behaviour is observed for inertial waves [20]. From here a conclusion may be drawn that the statement of the problem presented here within the framework of the Boussinesq model allows to describe inertial waves while the Brunt–Vaisala buoyancy frequency corresponds to the Coriolis frequency.

Conclusions

The article presentes the statement and the solution of a new problem of wave propagation in stratified fluid generated by a vertical jet intruding into the fluid. The dispersion equation is derived and the analysis of phase and group velocities is carried out. The dependences of phase and group velocities on the wave length λ and the Brunt–Vaisala buoyancy frequency N are found.

It is shown that for real values of the angular frequency ω and the wave number k solutions in the class of traveling waves propagating in stratified fluid from the jet in the radial direction do not exist. Howerer, there are wave disturbances localized near and propagating along the jet.

From obtained results it follows that there are waves in the external medium which propagate near the jet surface and they were observed in the experiments of [18].

The group velocity is shown to be negative that is the energy is transported in the direction counter to that of wave propagation. Moreover, the energy transport in the jet is sufficiently less than in the external near jet region.

REFERENCES

1. Lamb, H., *Hydrodynamics*, Cambridge University Press, Cambridge, 1932.
2. Milne-Thomson, L. M., *Theoretical Hydrodynamics*, Macmillan and Co. LTD, London, 1960.
3. Bai Shi-I, *Theory of Jets*, GIFML, Moscow, 1960326 [in Russian].
4. Kochin, N. E., Kibel, I. A. and Roze, N. V, *Theoretical Hydromechanics. Part I*, GIFML, Moscow, 1963 [in Russian].
5. Advanced Mechanics of Fluids, H. Rouse, Ed., New York, John Wiley & Sons1959.
6. Schlichting, H., *Grenzschicht-Teorie*, Karlsruhe, Verlag G. Braun, 1965.
7. Gospodinov, P., Radev, S. and Penchev, I., Velocity Profiles and Form of a Laminar Jet in Immiscible Liquid – Liquid Systems, *Int. J. Multiphase Flow*, 1979, 1, N 1, 87–99.
8. Needhan, D. I., Ridley, N. and Smith, J. H. B., A Jet in Crossflow, *J. Fluid Mech.*, 1988, 188, 159–184.
9. Ting Lu and Keller, J. B., Slender Jets and Thin Sheets with Surface Tension, *SIAM. J. Appl. Math.*, 1990, 50, N 6, 1533–1546.
10. Richards, J. R., Beris, A. N. and Lenhoff, A. M., Steady Laminar Flow of Liquid – Liquid Jets at High Reynolds Numbers, *Phys. Fluids*, 1993, A 5, N 7, 1703–1717.
11. Richards, J. R., Lenhoff, A. M. and Beris, A. N., Dynamic Breakup of Liquid – Liquid Jets, *Phys. Fluids*, 1994, 6, N 8, 2640–2655.
12. Longmire, E. K., Webster, D. R., Gefron, D. L. and Lowengrub, J. S., Dynamics of Pinch-off in Liquid Jetsflowing into Immiscible Liquid Ambients, *University of Minnesota. Supercomputer Inst. Res*, Report. UMSI 98/101, May, 1998, 7.
13. Yuen, M. C., Non-Linear Capillary Instability of a Liquid Jet, *J. Fluid Mech*, 1968, 33, Pt. 1, 151–163.
14. Mackova, M. P. and Shkadov, V. Ya., On Nonlinear Development of Capillary Waves in Jet of Fluid, *Izv. AN SSSR. MZhG*, 1972, N 3, 30–37 [in Russian].
15. Christiansen, R. M. and Hixson, A. N., Breakup of Liquid Jet in a Denser Liquid, *Ind. Eng. Chem.*, 1957, 49, 1017–1024.
16. Tomotika, S., On the Instability of a Cylindrical Thread of a Viscous Liquid Surrounded by Another Viscous Fluid, *Proc. R. Soc. London, Ser. A*, 1935, 150, 322–337.
17. Eroglu, A. and Breidenthal, R. E, Exponentially Accelerating Jet in Crossflow, *AIAA J.*, 1998, 36, N 6, 1002–1009.
18. Huq, P., Observations of Jets in Density Stratified Crossflows, *Atmospheric Environment*, 1997, 31, N 13, 2011–2022.
19. Pratte, B. D. and Baines, W. D., Profiles of the Round Turbulent Jets in a Crossflow, *J. Hydraulic Division, ASCE*, 1967, 93, N 1, 53–64.

20. Brekhovskih, L. M. and Goncharov, V. V., *Introduction to Mechanics of Continuums*, Nauka, Moscow, 1982 [in Russian].

21. Selezov, I. T. and Huq, P., Asymptotic-Heuristic Analysis of Nonlinear Water Wave Propagation in Two- and Three-Layer Fluids, *State Marine Techn. Univ*, St.-Petersburg, 1997, 237–244.

22. *Handbook of Mathematical Functions*, M. Abramowitz and I. A. Stegun, Eds., National Bureau of Standards Applied Mathematics, Series 55, 1964.

23. Smirnov, V. I., *Course of higher mathematics, Vol. 3, Part 2*, GIFML, Moscow, 1958 [in Russian].

24. Vaganov, R. B. and Kazenelenbaum, B. Z., *Basics of the Diffraction Theory*, Nauka, Moscow, 1982 [in Russian].

25. Mitropol'sky, Yu. Z., *Dynamics of Internal Gravitational Waves in the Ocean*, Gidrometeoizdat, Leningrad, 1981 [in Russian].

26. *Tables of Zeroes of Bessel's Functions*, BMT, Moscow, Computational Center of the USSR Acad. Sci., 1967, [in Russian].

27. Yanke, E., Emde, F. und Lösch, F., *Tafeln Höherer Funktionen*, B. G. Teubner Verlagsgesellschaft, Stuttgart, 1960.

□◆◆◆□