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Vortex fields and the Lamb–Stokes dissipation relation of fluid dynamics

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ABSTRACT

Energy dissipation in Newtonian fluids containing a unified vortex field is shown to depend on $-\eta \int_V (\varpi^2 + \lambda^2 \zeta^2)\, dV$, where η, ϖ and $\zeta = u \times \varpi$ are viscosity, vorticity and swirl. This term augments viscous dissipation where stream tube geometry is curved, e.g., in turbulent or helical flows.

1. Introduction

We present a physically motivated analysis showing how a *unified vortex field* contributes to the excitation and energy dissipation in fluid flows. The embedding of a unified vortex field, vortex field for short, into the description of a flow leads to new pathways for energy dissipation that are important in the description of some experimental flows. These new pathways augment the viscous dissipation described by the Navier–Stokes equations where such dissipation is proportional to velocity gradients. The physics describing these vortex field modes of dissipation are shown to extend the Navier–Stokes equations in a way that self-consistently modifies the velocity, vorticity, and swirl fields of the flow. Fluid flows can then be described by combining the Navier–Stokes equations and the equations for the vortex field.

The increased dissipation is also expected to be important for flows in which there are geometric constraints or coupling to external fields as occurs in astrophysical objects such as galaxies, stars, etc., magneto-hydrodynamic (MHD) and aeronautical flows [1]. This is especially true for geometrically constrained flows containing helical or swirling components to their flow field [2]. A common approach to describe such additional dissipation in a turbulent flow is to introduce flow-dependent eddy viscosities involving additional viscous stress terms into the Navier–Stokes equations [3]. The vortex field is an alternative to eddy viscosity approaches and has the additional advantage that it leads to a self-consistent fluid flow gauge theory. Another contemporary avenue of turbulence research consists of studying vortices in hydrodynamic flows [4]. The present approach differs in that we consider a unified vortex field made self-consistent with the Navier–Stokes equations.

In Sections 2–4 we describe how the embedding of the unified vortex field $\{\varpi, \zeta\}$ to the flow contributes to the balance of stress-energy in the flow. We show how this generates an additional term in the Lamb–Stokes (kinetic) energy dissipation relation that we call the λ-effect. It is shown how the inclusion of the λ-effect leads to a modification of the Navier–Stokes equations. (This modification vanishes in the absence of the vortex field.) The λ-effect also leads to a physically appealing explanation of the long-standing problem of vortical structure and augmented energy dissipation in helical pipe flows measured by White and Taylor [5–7].

2. Vector fields characterizing a flow

In a three-dimensional vector space, given non-collinear velocity u and vorticity ϖ vectors, we can form another vector $\zeta = u \times \varpi$, called the swirl vector. (In this, the negative of the vector ζ is sometimes called the Lamb vector [8].) From the last two vectors, the *unified vortex field* combines the vorticity, $\varpi \approx \nabla \times u$, *and* the swirl, $\zeta \approx u \times \varpi$, field vectors in a way that is similar to the unification of the electric and magnetic fields forming the electromagnetic field. This is important because the unified vortex field $\{\varpi, \zeta\}$ makes a contribution to the stress-energy balance [9, p. 505] in fluid flows that is additional to the stress-energy accounted by the Navier–Stokes equations [10]. The stress-energy in the case of Navier–Stokes theory combines the energy of the flow with the stress tensor in a way that respects the finite speed of propagation of transverse waves in the flow [11]. From the kinetic energy part of the stress-energy, proportional to u^2, we compute the energy dissipation $d\mathcal{E}/dt$ to be proportional ϖ^2 as shown in the next section. Thus, the Navier–Stokes stress-energy is a function of u (and its spatial derivatives) and its energy dissipation has

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a dependence on ϖ. The characterization of the flow by the vectors $\{u, \varpi, \zeta\}$ can be further linked to physical processes in the flow.

For wave systems in a fluid constrained by a maximum speed, c_m, propagation is necessarily accompanied by a transfer of energy–momentum [9, p. 505]. This energy–momentum transfer can occur via transverse vortex wave propagation or longitudinal wave propagation, in addition to the stress-energy derived from the Navier–Stokes equations. Finite speed, longitudinal sound waves travelling in a moving fluid lead to energy–momentum transfer in any direction. Thus, this transfer could be parameterized by all of the vectors $\{u, \varpi, \zeta\}$. In Navier–Stokes theory, for compressible fluids, longitudinal sound waves have finite speeds, whereas for incompressible fluids this speed is infinite. The propagation speed, c_m, is finite for transverse waves. As shown below, finite speed vortex $\{\varpi, \zeta\}$ wave fields can be effective carriers of the energy–momentum in addition to the velocity field u, since their effect spans all length scales and they couple into the flows's inertia. We will also show that the vortex field's stress-energy depends on ϖ^2 and ζ^2. As these fields contribute a stress-energy that is additional to that contributed by the Navier–Stokes theory, we say the vortex field represents an *excitation* of the flow. The excitations lead to an increase in the *energy dissipation* (rate) $d\mathcal{E}/dt$. Thus, the stress-energy balance of the flow including the vortex field, the fluid inertia, and the Navier–Stokes stress-energy can be expressed as a function of the three directions $\{u, \varpi, \zeta\}$ that physically characterize the flow. Then, in the description of the stress-energy balance of the flow involving the inertial, viscous, and vortex fields, each component can be expressed as a function of the three vectors $\{u, \varpi, \zeta\}$.

3. Lamb–Stokes dissipation relation

For an incompressible fluid described by the Navier–Stokes equations, we obtain an estimate of the energy dissipation. This allows us to introduce the Lamb–Stokes energy dissipation relation. Energy (dissipation) relations are important because they have the easiest to identify vortex field contribution. The Navier–Stokes equations are written in the form [9]

$$\rho\left(\frac{\partial u}{\partial t} + u\cdot\nabla u\right) + \nabla p = \eta\nabla^2 u, \tag{1}$$

$$\rho\frac{Du}{Dt} + \nabla p = \eta\nabla^2 u, \tag{2}$$

$$\nabla\cdot u = 0. \tag{3}$$

The left-hand side describes the inertial effects of the flow. It contains the material derivative of the Eulerian velocity field as well as the effects of a pressure gradient. The convective term $u\cdot\nabla u$ of the total derivative D/Dt contains the main non-linearity leading to the development of turbulence. In the standard notation, ρ is the fluid density, the velocity vector is denoted by u, the pressure is given by p, the absolute viscosity is η. (In the fluid mechanics literature, the absolute viscosity [gm/cm s] is often denoted μ. We use the notation of Ref. [9].) The pressure gradient, ∇p, accounts for the work needed to maintain the flow against viscous losses found on the right-hand side. For an incompressible fluid, Eq. (3) is the equation of continuity.

To compute the rate of energy dissipation of a fluid described by Eqs. (1)–(3), consider first the kinetic energy given by,

$$\mathcal{E} = \frac{1}{2}\int\limits_{\Delta}\rho u^2\,dV, \tag{4}$$

where $u^2 = u^i u_i$ is the squared magnitude of the Eulerian velocity field whose components in the three orthogonal coordinate directions are given by u^i. To determine the energy dissipation $d\mathcal{E}/dt$ from Eqs. (1) and (4) [12, pp. 580–581], [13], use Eq. (1) and take the dot product with u, then integrate over a volume Δ. This gives

$$\frac{d\mathcal{E}}{dt} \equiv \frac{1}{2}\int\limits_{\Delta}\rho\frac{Du^2}{Dt}dV + \int\limits_{\Delta}u\cdot\nabla p\,dV = \eta\int\limits_{\Delta}u\cdot\nabla^2 u\,dV. \tag{5}$$

The pressure integration term $\int u\cdot\nabla p\,dV = 0$ is eliminated when we consider the virtual work of the pressure over a differential volume for an incompressible fluid ($\nabla\cdot u = 0$) when $u\cdot n = 0$ or the surface is at infinity [12, p. 580, Eqs. (4)–(6)]. We next use the identity $\nabla\times(\nabla\times A) = \nabla(\nabla\cdot A) - \nabla^2 A$ applied to the velocity field for an incompressible fluid ($\nabla\cdot u = 0$) obtaining for the integrand in the last term on the right-hand side of Eq. (5) the expression $u\cdot\nabla^2 u = -u\cdot(\nabla\times\varpi)$. We then substitute this into Eq. (5) obtaining

$$\int\limits_{\Delta}\frac{1}{2}\rho\frac{Du^2}{Dt}dV = -\eta\int\limits_{\Delta}u\cdot(\nabla\times\varpi)\,dV. \tag{6}$$

Next apply the identity $\nabla\cdot(A\times B) = B\cdot\nabla\times A - A\cdot(\nabla\times B)$ using $A\to u$ and $B\to\varpi$ so that $u\cdot(\nabla\times\varpi) = \varpi\cdot(\nabla\times u) - \nabla\cdot(u\times\varpi)$. Using this in Eq. (6) yields in succession

$$\begin{aligned}\frac{1}{2}\int\limits_{\Delta}\rho\frac{Du^2}{Dt}dV &= -\eta\int\limits_{\Delta}\big(\varpi\cdot(\nabla\times u) - \nabla\cdot(u\times\varpi)\big)\,dV \\ &= -\eta\int\limits_{\Delta}\varpi^2\,dV + \eta\int\limits_{\partial\Delta}(u\times\varpi)\cdot dS \\ &= -\eta\int\limits_{\Delta}\varpi^2\,dV + \eta\int\limits_{\partial\Delta}\zeta\cdot dS. \end{aligned}\tag{7}$$

Stokes' theorem was used in the transition of the first to the second line to convert the last volume integral into a surface integral. Using the continuity equation, $D\rho/Dt = (\partial\rho/\partial t + u\cdot\nabla\rho) = 0$, on the left-hand side, and the definition in Eq. (5), there is obtained the *Lamb–Stokes energy dissipation relation*

$$\frac{d\mathcal{E}}{dt} = -\eta\int\limits_{\Delta}\varpi^2\,dV + \eta\int\limits_{\partial\Delta}\zeta\cdot dS. \tag{8}$$

The first term on the right-hand side shows the energy dissipation (rate) is proportional to the second moment of the vorticity field. The second term gives the work of surface traction over the boundary $\partial\Delta$ of the volume Δ. In these equations, we have not introduced a factor of 1/2 in the definition of the vorticity. If we had, the surface integral would have an extra factor of 2 inserted [12, p. 581], [13]. The normals point outward. The absolute viscosity has units $[\eta] = E\times T/L^3$. The quantity $\eta\int_\Delta\varpi^2\,dV$ then has units of $[\eta][\varpi^2][dV] = E\times T/L^3 T^{-2} L^3 = E/T$.

4. Vortex fields

There are other possible avenues of energy dissipation in a turbulent fluid flow that are not reflected in Eq. (8). These are associated with vortex flow structures involving the swirl ζ and vorticity ϖ vectors leading to transverse waves that transport energy in a direction perpendicular to the (ϖ, ζ)-planes. The vortex field can have a wide range of wavelengths which couple to the fluid leading to additional short range dissipation and long range convection.

Further insight into the structure of vortex waves in fluid dynamics can be obtained by considering an analogy with the theory of transverse waves in electromagnetism [14]. In an electromagnetic field problem the energy dissipation, expressed as the production of Joule heat, is given by the time derivative of the integral $\mathcal{I} = \frac{1}{2}\int_\Delta\frac{1}{4\pi}(E^2 + B^2)\,dV$. (See [15, p. 79].) Here E is the magnitude of the electric field and B is the magnitude of the magnetic induction field. In electromagnetics the integral $\mathcal{I}$ is proportional to the time component of the Maxwell stress-energy tensor. The viscosity coefficient η converts the analogous integral in Eq. (8) to an energy dissipation rate without requiring differentiation. Here, the B-field is analogous to the vorticity ϖ and the E-field is analogous to the swirl field ζ. Thus, we would expect the Lamb–Stokes dissipation relation, Eq. (8), in order to include a vortex field, to have the term $\lambda^2\zeta^2 = \lambda^2(u\times\varpi)^2$ contributing to the total dissipation. In the case of a fluid, the coupling of the transverse waves to the fluid momentum hinders the propagation of the waves, so such a wave is likely to be short ranged, i.e., evanescent, unless energy is continually supplied to sustain the vortex.

We now consider consequences of combining the vortex field $\{\varpi,\zeta\}$ and the vorticity ϖ originating in the Navier–Stokes equations. The vortex wave system consists of both standing, evanescent, and travelling vortex waves. The presence of vortex waves alters the balance of dissipation in a region as the vortex energy–momentum can enter or leave a region or be created in a region. Since the vorticity satisfies a diffusion equation, as can be seen by taking the curl of Eq. (1), the vortex waves, have a diffusive character as well as a propagating component. The vortex field contributes an energy density $-\frac{\eta}{2}(\varpi^2+\lambda^2\zeta^2)$ to the total dissipation. Thus, in a fluid dynamical system, the total energy dissipation rate depends not only on $-\eta\int_\Delta \varpi^2\,dV$ but also upon $-\eta\int_\Delta \lambda^2\zeta^2\,dV$. On the other hand, from the Navier–Stokes equation we have a term $-\eta\int_\Delta \varpi^2\,dV$, as shown above, that accounts for the total Navier–Stokes dissipation when the vortex waves are not considered part of the flow. The combination of viscous dissipation due to the Navier–Stokes equation and the vortex dissipation can then be analyzed by partitioning the total dissipation and rescaling λ since $\frac{\eta}{2}\varpi^2 + \frac{\eta}{2}(\varpi^2+\lambda^2\zeta^2) = \eta(\varpi^2+\frac{1}{2}\lambda^2\zeta^2) \equiv \eta(\varpi^2+\lambda^2\zeta^2)$, where $\lambda = \sqrt{2}\lambda$. This divides the dissipation into $-\frac{\eta}{2}\varpi^2$ from the Navier–Stokes part and $-\frac{\eta}{2}(\varpi^2+\lambda^2\zeta^2)$ from the vortex part with a total $\eta(\varpi^2+\lambda^2\zeta^2)$ as described by the *generalized* Lamb–Stokes dissipation relation:

$$\frac{d\mathcal{E}}{dt} = -\eta\int\limits_{\Delta}(\varpi^2+\lambda^2\zeta^2)\,dV + \eta\int\limits_{\partial\Delta}\zeta\cdot dS. \tag{9}$$

The *generalization* is to include the term $\lambda^2\zeta^2$ which is needed for describing dissipation in flows having a vortex field generating helical, transverse waves of various wavelengths. The coefficient λ is a material parameter, specific to each fluid, having units $[\lambda^2] = T_0^2/L_0^2$. These are the same units as $1/c_m^2$ where c_m is the maximum speed of propagation of a longitudinal or a transverse vortex wave in a fluid medium (i.e., sound). The new term represents an additional mechanism of dissipation given by $\lambda^2\zeta^2 = \lambda^2(u\times\varpi)^2$—we call this the λ-*effect* term. If energy dissipation is described by relative slippage of fluid lamella, the parametrization of enhanced energy dissipation in Eq. (9) is given by the λ-effect when the slippage includes transverse waves or rotation in addition to shear slippage. The weighted sum of the terms $-\eta\int_\Delta(\varpi^2+\lambda^2\zeta^2)\,dV$ and $-\eta\int_\Delta(\varpi^2)\,dV$ is interpreted as the dissipation due to a vortex wave field in addition to the background dissipation described by the Navier–Stokes equations. By combining the vortex field with the description of the flow based on the Navier–Stokes equations, we can determine how the field changes the Navier–Stokes equations.

The added physics due to the vortex field can be further understood by examining the role of $\zeta \approx u\times\varpi \approx u\times(\nabla\times u)$ as an agent of momentum transfer in the Navier–Stokes equations. Let us consider the Navier–Stokes equation's convective derivative part in Eq. (1):

$$\rho\left(\frac{\partial u}{\partial t} + u\cdot\nabla u\right) = \cdots. \tag{10}$$

Then, using the identity, $\frac{1}{2}\nabla u\cdot u = (u\cdot\nabla)u + u\times(\nabla\times u)$, and setting $\zeta \approx u\times\varpi$, and $\varpi \approx \nabla\times u$, it is found that the convective derivative part can be written as

$$\rho\frac{\partial u}{\partial t} - \rho\zeta + \rho\nabla\left(\frac{1}{2}u^2\right) = \cdots. \tag{11}$$

Since $\zeta \approx u\times\varpi \approx u\times(\nabla\times u)$ appears on the left-hand side of Eq. (11), both the total inertia and the λ-effect contain the "centrifugal" term $u\times(\nabla\times u)$ that convects the flow especially in curved geometries. Using Eq. (11) and adding a linear dependence on the swirl vector $\rho(s\lambda c_m - 1)\zeta^i$ accounting for the vortex field to the balance of stress on the right-hand side that includes the viscous stress term $\frac{\eta}{\rho}\nabla^2 u^i$ and minus the pressure gradient, leads to a modified Navier–Stokes equation in a form that displays the physics involved:

$$\frac{\partial u^i}{\partial t} + \nabla^i\left(\frac{1}{2}u^2\right) \approx -\frac{1}{\rho}\nabla^i p + s\lambda c_m\zeta^i + \frac{\eta}{\rho}\nabla^2 u^i. \tag{12}$$

It is shown below that numerical factor $s\to 1$. Eq. (12) can also be written for direct comparison to Eq. (1) as

$$\frac{\partial u^i}{\partial t} + u\cdot\nabla u^i = -\frac{1}{\rho}\nabla^i p + (\lambda c_m - 1)\zeta^i + \frac{\eta}{\rho}\nabla^2 u^i. \tag{13}$$

Returning to Eq. (12), we see that the term, $s\lambda c_m\zeta^i$, is a combination of the vortex field and a term coming from the inertia of the flow. The units of c_m are the units of the speed of the transverse waves. With the factor of c_m the units of all terms in Eq. (12) are $[L/T^2]$. While a linear dependence on ζ is the simplest assumption to account for the additional stress due to the vortex field, it can be derived from the full theory [16]. The swirl field or λ-effect term $\lambda c_m\zeta^i$ is responsible for the centrifugal-inertial effects in the flow leading to secondary vortical flows, e.g., in helical pipe flows [17] and other cases of curved stream tubes. The other terms in Eq. (12) for time independent flow can be analyzed as follows. The term $\nabla^i(\frac{1}{2}u^2)$ is the divergence of the so-called dynamic head or dynamic pressure, the pressure term $-\frac{1}{\rho}\nabla^i p$ contains the driving pressure gradient, and the viscous term $\frac{\eta}{\rho}\nabla^2 u^i$ acts to diffuse the velocity components of the flow.

The "classical" limit as $\lambda\to 0$ requires $c_m\to\infty$ in a way that $\lambda c_m\to 1$. This is similar to the limit of infinite speed of longitudinal sound waves for the Navier–Stokes equations with zero compressibility. This limit also implies that the factor $s=1$ in Eq. (12). We would not include a similar linear contribution of ϖ in Eq. (12) as this is already included in the dissipation as shown in Eqs. (8). Here we see in Eq. (12) that the λ-effect modifies the nonlinear convective term that appears in the total derivative of Eq. (2) since $\lambda c_m\zeta^i \approx \lambda c_m u\times(\nabla\times u)$ so that the λ-effect affects all scales of the flow.

According to Eq. (12), when we include the unified vortex field, the λ-effect is always present. So, in geometries where the swirl vector ζ is important, as in Eq. (12), it will change the flow, and lead to quantitative changes to velocity profiles. When the λ-effect reduces the $u\times(\nabla\times u)$ term of the fluid inertia, it delays the development of turbulence in transitional flows. The λ-effect suppresses instabilities and small wavelength turbulence in the flow leading to increased values of the critical Reynolds number. The surprise is that Eq. (12) appears to be a Navier–Stokes equation, but it describes a different balance of dissipative and inertial effects because $\lambda c_m \neq 1$.

We can show how the diffusion and convection of vorticity is modified by taking the curl of Eq. (12) using $\varpi\approx\nabla\times u$, $\zeta\approx u\times\varpi$

and the identity $\nabla \times (u \times \varpi) = (\varpi \cdot \nabla)u - (u \cdot \nabla)\varpi + u(\nabla \cdot \varpi) - \varpi(\nabla \cdot u)$. There is obtained an equation for vorticity convection and diffusion

$$\frac{\partial \varpi^i}{\partial t} + \lambda c_m (u \cdot \nabla \varpi - \varpi \cdot \nabla u) = \frac{\eta}{\rho} \nabla^2 \varpi^i. \tag{14}$$

The terms on the left of Eq. (14) give the convective derivative of the vorticity. Thus the vorticity satisfies a convective diffusion equation. This equation then shows that vorticity convection is scaled by λc_m and that vorticity is not conserved because of the diffusional term $(\eta/\rho)\nabla^2 \varpi^i$. Thus, the vortex field $\{\varpi, \zeta\}$ will be changed by the λ-effect when it is self-consistently included into the calculation of the flow. This implies that the swirl is similarly affected as it is part of the unified vortex field.

The general vortex field theory of a compressible viscous fluid [16] shows the vortex field $\{\varpi, \zeta\}$ has a vector potential A_μ. It is coupled self-consistently to the fluid flow through an inhomogeneous wave equation with the fluid current as a source. This shows that the unified vortex field exists since it is derivable from the vector potential. The formulation of the general theory is invariant under so-called acoustic Lorentz transformations, thus satisfying the most basic physical requirements of causality in a medium where the greatest speed of propagation of a disturbance is the maximum speed of the transverse sound wave c_m. This allows one to show that there is a map (homeomorphism) between electromagnetic field theory and the general vortex theory justifying the analogy above. A fluid flow then consists of a Navier–Stokes flow driven by unified vortex field excitation derived from a self-consistent vector potential. Turbulent flows then contain a good measure of vortex excitation.

Depending on the swirl vector $\zeta \approx u \times \varpi$, the λ-effect may be small in many flow geometries but large in some particular ones. Thus, there are natural geometries in which the λ-effect is enhanced. These geometries are generally helical. But local stream tubes with curved geometries also play a role in turbulence [2]. An example of such a geometry occurs in a 3D helical pipe flow [18, 19]. White and Taylor's measurements [5,6] of helical pipe flow showed strongly enhanced energy dissipation involving the creation of secondary vortical flows [17], [20, p. 567]. The effects are striking—in these measurements of helical pipe flow, the enhanced dissipative loss, as measured in terms of head loss, is as large as 600% compared to head losses in straight pipe flow. The effect is so pronounced that turbulent pipe flow in a straight pipe section (with Reynolds numbers above 2300 but below about 5000) appears to "relaminarize" on entry to a helical pipe section [21–24]. This effect can be alternatively characterized as the development of a non-chaotic turbulent (vortex) structure. Thus, the enhanced dissipation due to the vortex field (λ-effect) in the Lamb–Stokes dissipation relation and the scaling of the centrifugal effects in the convection of inertia provide a physically appealing explanation of the experimental results of White and Taylor and introduces a new mechanism for describing the development of turbulent structures.

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