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## Evolution of helicity in fluid flows

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An invariant helicity integral and a differential helicity evolution equation are found for *viscous* fluid flows. A geometrodynamical approach is used, which includes a vortex field. The vortex field is derivable from a vector potential  $A$ . The vector potential is then used to characterize the evolution of flow topology. The source of the helicity is found to be the topological parity  $k=2\lambda\omega\cdot\zeta$  and the moving boundary surfaces of the fluid. Here,  $\omega$  and  $\zeta$  are the vorticity and swirl components of the vortex field  $\{\omega, \zeta\}$  and  $\lambda$  is a constitutive or material parameter of the fluid. Our first result using the vector calculus identifies the scalar helicity as  $h_t=A\omega$ . This result is then generalized using the calculus of differential forms, yielding other results including the existence of a helicity current vector proportional to  $(\phi\omega-\lambda A\times\zeta)$ . © 2010 American Institute of Physics. [doi:10.1063/1.3329422]

### I. INTRODUCTION

Helicity provides a measure of the propensity of flows to form vortices or coherent structures. This measure was first developed by Woltjer for electrodynamics (Woltjer, 1958). A theory of helicity and the topology of *inviscid* or perfect fluid flows was subsequently developed (Moffatt, 1969; Levich and Tsinober, 1983; Berger and Field, 1984; Moffatt and Ricca, 1992; Moffatt and Tsinober, 1992; and Branover *et al.*, 1999). For viscous fluids, this inviscid fluid theory of helicity applies only approximately. The goal of the present work is to remove this limitation. This is achieved by developing a new, related, definition of helicity for viscous fluids using geometrodynamical methods (Nakahara, 2003; Flanders, 1989; and Misner *et al.*, 1973). By using this new definition, much of the previous work on the topology of perfect fluids, e.g., summarized in Ricca (2009) and Ricca (2001), can now be extended to viscous fluids and experimentally tested. Significantly, the existence of a vector potential in our theory allows one to formulate rigorous measures of the topological evolution of the excitations in a fluid (Kiehn, 2007). The formulation developed here also yields an exact association of topological change in viscous fluids flow to the avenues of energy dissipation caused by the presence of the fluid flow's vortex field.

The theory of helicity developed here uses the theory of geometrofluid dynamics (GFD) (Scofield and Huq, 2008; Scofield and Huq, 2009) based on the exterior calculus. In GFD the stress-energy flux balance of the fluid includes the presence of a *vortex field*, a fluid analog of the electromagnetic field. In GFD, transverse waves are excitation modes of the vortex field *strongly coupled* to the flow. These vortex field modes not only change the stress-energy flux balance by their presence; they also change the topology and energy dissipation of the flow. The physical existence of the fluid vortex field has been experimentally corroborated as discussed in Scofield and Huq (2008; 2009). The GFD theory has some of the structure of a relativistic fluid theory (Misner *et al.*, 1973, Sec. 22.3; Landau and Lifshitz, 1959, Chap. XV) in which the maximum speed of transverse waves is reduced to  $c_m$  from the speed of light and the vortex field has the

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differential geometrical and topological structure of an electromagnetic field complete with a vector potential.

The GFD consistently includes  $c_m$  by using a (special) acoustic space-time (SAST). This space-time is a special case of more general acoustic space-times (Unruh, 1995; Barceló *et al.*, 2001; and Schützhold and Unruh, 2005). Such space-times are of interest as analogs of the curved space-times occurring in general relativity [see Barceló (2005) for a recent review]. The present work thus raises the possibility of experimentally accessible space-time analogs. Such acoustic space-times previously were based on a longitudinal wave analogy. In the present development, the fluid transverse waves are closer to transverse gravitational waves for the purposes of analogy.

This paper is organized as follows. In Sec. II, we compare and contrast two methods of characterizing fluid topology, examine various formulations of helicity, and survey the GFD needed for its calculation. In Sec. III we develop the expression for the viscous fluid helicity invariant and its evolution. A summary and conclusions section follows.

## II. BACKGROUND

This section first describes two topological methods for characterizing a fluid flow. Then, it discusses several alternate definitions of helicity and introduces our definition of helicity,  $\mathcal{H}_G$ . The geometrofluid dynamical theory of fluids is then described as it supplies the vector potential needed for the evaluation of  $\mathcal{H}_G$ .

### A. Topological measures of flow

The ubiquitousness of helical structures leading to turbulence in fluid flow continues to be a persistent research topic (Branover *et al.*, 1999). A signature of the transition to turbulent flows is topology change. Thus, the ability to quantify measures of fluid topology and topological change is important in understanding the development of turbulence and other aspects of fluids and their flow. There are two broad measures of the topology of such flows. The first is helicity  $\mathcal{H}_G$ , which measures the intertwining of flow tubes. As shown by the helicity theory of perfect fluids referenced above, a helicity measure is related to the knotting and linking of the flow tubes among each other. The knotting and linking of a flow manifold provides a topological classification of such manifolds (Nakahara, 2003; Rolfson, 2000). The second measure is the Euler characteristic  $\chi$ , which provides a measure of the number of substructure elements in the flow and is consequently another means for topological classification of flow manifolds. For instance, the Euler characteristic of a handle body is related to its genus  $g$  (the number of holes) by  $\chi=2-2g$ . As the genus of the flow manifold varies, so too will the knottedness and linking of the flow. An advantage of these measures is that  $\mathcal{H}_G$  is an integral measure and  $\chi$  has an integral measure that can be obtained from the  $n$ -dimensional Gauss–Bonnet theorem (Nakahara, 2003). These measures are insensitive to small geometric changes in the flow and therefore provide a description of the persistence of structure in a flow. This is the sense of a topological invariant. Significantly, because topology preserving numerical integrators for vortex field coupled systems have been derived (Desbrun *et al.*, 2005a; 2005b), one can follow the topological change in computed fluid system evolution. Thus, the theory of knotted and linked manifolds (Rolfson, 2000; Ren *et al.*, 2007) can be effectively applied to such computations. Changes in either  $\mathcal{H}_G$  or  $\chi$  signal topology change in the flow. Surprisingly, the Euler characteristic is also a sensitive measure of the topology changes in a classical configuration space occurring in thermodynamic phase transitions of, e.g., a fluid (Franzosi *et al.*, 2000; 2007; and Franzosi and Pettini, 2007). Their work provides tools for examining the large  $N$  or many-body limit of the evolution of topological structure in a fluid. Thus, topological methods find use from the interacting particle-level models of fluids in phase transitions to the dynamics of fluid flows. Here, the focus is on the helicity measure for viscous fluids,  $\mathcal{H}_G$ , primarily because of the need for a topological fluid flow theory that can be applied to real viscous fluids.

## B. Helicities

The notion of helicity was first defined for electromagnetic field theory in one temporal and three spatial dimensions  $(ct, x, y, z)$ . Here, the magnetic helicity is given by the volume integral (Woltjer, 1958)

$$\mathcal{H}_W = \int_{\Delta} A \cdot B d^3x, \quad (1)$$

where  $A$  is the electromagnetic vector potential and  $B = \nabla \times A$  is the magnetic field.  $\mathcal{H}_W$  is gauge and time transformation invariant but not parity invariant ( $B$  is a pseudo-vector) in a space-time. The importance of such invariants in fluid dynamics was recognized by Moffatt as being applicable to the dynamics of perfect fluids. He introduced the integral (Moffatt, 1969)

$$\mathcal{H}_M = \int_{\Delta} u \cdot \Omega d^3x. \quad (2)$$

This can be understood from the fact, given the velocity  $u$ , and the following Euler–Navier–Stokes relation for the vorticity

$$\Omega \equiv \nabla \times u, \quad (3)$$

that  $u$  is like the vector potential for  $\Omega$ . This conclusion is supported by the fact that in the Euler–Navier–Stokes theory,  $\Omega$  is like the magnetic field in that  $\nabla \cdot \Omega = \nabla \cdot \nabla \times u = 0$ , so Eq. (3) above can be inverted as

$$u^i = \epsilon_{jk}^i \int \Omega^j \partial^k G(x, y) d^3y. \quad (4)$$

Here,  $\epsilon_{jk}^i$  is the completely antisymmetric Levi-Civita symbol in three-dimensional (3D) and

$$G(x, y) = \frac{1}{4\pi} \frac{1}{\|x - y\|} \quad (5)$$

is the Green's function for the Laplacian. This expression and the one before it show that there is a potential for the velocity field for a perfect fluid so that Eq. (2) is a fluid analog to Eq. (1).

The objective of the present paper is to extend the above ideas of helicity in which there are transverse wave excitations changing the topology of viscous fluid flow. This definition, in terms of differential forms, is a Hopf invariant integral generalizing Eq. (1),

$$\mathcal{H}_G = \frac{1}{16\pi^2} \int_{M^3} A \wedge dA. \quad (6)$$

For this extension, GFD is needed as it supplies the vector field  $A$ .

For reference, the situation for perfect fluids is as follows. The velocity  $u$  is the potential of the vorticity. A definition of helicity conservation as  $u \wedge du = 0$  is tantamount to Frobenius integrability. The condition of Frobenius integrability in 3D for the velocity field is that  $u \times (\nabla \times u) = 0$ . If we set the vorticity  $\Omega = \nabla \times u$  (as we can in the Euler theory of perfect fluids), when the velocity field is not Frobenius integrable, helicity can be produced. Contrarily, the helicity density  $h = u \cdot (\nabla \times u)$  is conserved ( $\nabla \cdot h = 0$ ) for a perfect fluid and it does not change. In this sense, changes in the nonorthogonality of the vectors  $u$  and  $\Omega$  for a viscous fluid provide a measure of helicity production. Thus, the helicity density is a measure of how much the velocity and the vorticity are not orthogonal  $h = u \cdot \Omega = \|u\| \cdot \|\Omega\| \cos \theta_{u\Omega}$ . It is interesting to note that in two-dimensional viscous flow we can define as usual  $\Omega = \nabla \times u$ , then the velocity Frobenius condition is satisfied. Furthermore, in this case, there is a stream function for the velocity. However, such a formulation does not yield a vector potential.

### C. GFD flow equations

GFD is based on the realization that fluid flows can be described in terms of excitations of a base flow. There are just two kinds of excitation modes in a homogeneous isotropic fluid without a free surface. They are longitudinal and transverse waves. Longitudinal sound waves are generally of small amplitude and do not change the topology of the flow. Small amplitude short wavelength, high frequency transverse waves are generally evanescent. However, long wavelength, low frequency transverse waves can be strongly coupled to the flow and be long lasting because of this coupling. As we show, these transverse wave modes of the vortex field can change the topology of the flow.

In GFD the vorticity vector  $\omega$  and swirl vector  $\zeta$  field components of the vortex field  $F$ , together with its vector potential  $A$  and current  $J = \rho u$ , are the main physical fields. These quantities are constrained by stress-energy flux balance equations that, for slow flow, limit to the Navier–Stokes equations (NSEs) (Scofield and Huq, 2009). This is in contrast to Navier–Stokes viscous fluid and Euler perfect fluid theories where the vorticity and swirl are *defined* as  $\nabla \times u$  and  $u \times \omega$ , respectively, so that these quantities do not enter into the stress-energy flux balance of the flow.

The formulation of the fluid dynamics by GFD differs from the Navier–Stokes theory (NST) because of the inclusion of the vortex field with its associated vector potential. This enables the unambiguous definition of the vector potential  $A$  and hence of the helicity scalar  $h_t = A \cdot \omega$ . Furthermore, the potential  $A$  allows one to compute the vortex field  $F = \{\omega, \zeta\}$ . The quantities  $\{\omega, \zeta\}$  do not take the same values as found from the Euler or NSEs, but have similar physical meaning, hence we adopt the older nomenclature. The vortex field  $F$  is therefore not the same as a field of vorticity.

The GFD flow equations describe a flow with transverse wave modes. Similar to the NSEs, GFD also express the balance of inertial and other stresses in a form derived from Newton’s law of viscosity. The NSE balance is formulated to describe laminar viscous flow in a Newtonian space-time. When additional dynamical *structures* such as occurs in turbulent flow having a vortex field, the Navier–Stokes stress balance needs to be modified since these additional structures carry energy or momentum (Landau and Lifshitz 1959, p. 509). The stress balance involves three different stress-energies in a SAST. These are inertial  $\tau_e^{\kappa\nu}$ , Newtonian viscous  $\tau_n^{\kappa\nu}$ , and vortex field  $\tau_m^{\kappa\nu}$ . The full set of equations needed to self-consistently determine the velocity, vorticity, swirl fields, and energies are given by the following (Scofield and Huq, 2009):

$$(\tau_e^{\kappa\nu} - \tau_n^{\kappa\nu} - \tau_m^{\kappa\nu})_{;\nu} = 0, \quad (7a)$$

$$(\tau_e - \tau_n)^{\mu\nu}_{;\nu} = -\dot{F}^{\mu\nu} j_\nu, \quad (7b)$$

$$\dot{H}_{\kappa\lambda} = C^{\mu\nu}_{\kappa\lambda} \dot{F}_{\mu\nu}, \quad (7c)$$

$$\dot{F}_{\mu\nu} = \frac{1}{2} \left( \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} \right), \quad (7d)$$

$$-\square A_\mu = -(\partial_t^2 - \nabla^2)A_\mu = \frac{4\pi}{\eta} j_\mu. \quad (7e)$$

Here,  $\tau_e^{\mu\nu} = \rho u^\mu u^\nu + p(g^{\mu\nu} + c_m^{-2} u^\mu u^\nu)$  is the Lorentz transformation invariant stress-energy of the mass distribution using a maximum transverse wave speed  $c_m$ . Here,  $p$  is the pressure and  $\rho$  is the density. The stress-energy  $4\pi \tau_m^{\mu\nu} = g^{\mu\alpha} \dot{F}_{\alpha\beta} \dot{F}^{\beta\nu} - \frac{1}{4} g^{\mu\nu} \dot{F}_{\alpha\beta}^* \dot{F}^{\alpha\beta}$  is a symmetric matrix that is traceless, and the Newtonian fluid stress-energy for a fluid with absolute viscosity  $\eta$  is given again by a symmetric matrix with components  $\tau_n^i = 2\eta(\tilde{\sigma}_{\mu\nu} + \delta\theta\mathcal{P}_{\mu\nu})$ . In the latter, the stress-energy projected into a spacelike 3-volume is given by  $\tilde{\sigma}_{\mu\nu} = \frac{1}{2}(\mathcal{P}_{\nu\mu;\epsilon}^\epsilon + \mathcal{P}_{\mu\nu;\epsilon}^\epsilon) - \frac{1}{3}\theta\mathcal{P}_{\mu\nu}$ , where the projection op-

erator is given by  $\mathcal{P}_{\mu\nu} = g_{\mu\nu} + \hat{u}_\mu \hat{u}_\nu$  in terms of the SAST metric  $(g_{\mu\nu}) = \text{diag}(-1, 1, 1, 1, \dots)$ ,  $c_m = 1$ , and the normalized velocity vector components  $\hat{u}_\mu$ . Additionally,  $\theta = u_{;\mu}^\mu$ . The 4-velocities  $u_\mu$  satisfy the normalization condition ( $c_m = 1$ ),  $u^\mu u_\mu = -1$ . The quantity  $\bar{\eta}$  scales the current and its units are chosen so that the wave equation, for the vector potential [Eq. (7e)] has consistent units. The parameters  $C_{\kappa\lambda}^{\mu\nu}$  are constitutive (material) parameters of the vortex fields that can be expressed in terms of four quantities  $\{\bar{\lambda}, \lambda, \kappa, \bar{\kappa}\}$  in the following manner [Eq. (7c)]:

$$(-\star \mathring{H}_{\kappa\lambda}) = \begin{pmatrix} 0 & -\bar{\lambda}\xi_1 & -\bar{\lambda}\xi_2 & -\bar{\lambda}\xi_3 \\ \bar{\lambda}\xi_1 & 0 & \bar{\kappa}\varpi_3 & -\bar{\kappa}\varpi_2 \\ \bar{\lambda}\xi_2 & -\bar{\kappa}\varpi_3 & 0 & \bar{\kappa}\varpi_1 \\ \bar{\lambda}\xi_3 & \bar{\kappa}\varpi_2 & -\bar{\kappa}\varpi_1 & 0 \end{pmatrix}, \quad (\mathring{F}_{\kappa\nu}) = \begin{pmatrix} 0 & -\lambda\zeta_1 & -\lambda\zeta_2 & -\lambda\zeta_3 \\ \lambda\zeta_1 & 0 & \kappa\omega_3 & -\kappa\omega_2 \\ \lambda\zeta_2 & -\kappa\omega_3 & 0 & \kappa\omega_1 \\ \lambda\zeta_3 & \kappa\omega_2 & -\kappa\omega_1 & 0 \end{pmatrix}, \quad (8)$$

so the components of  $\mathring{\tau}_m^{\mu\nu}$  are computed to be

$$4\pi \mathring{\tau}_m^{00} = \frac{1}{2}(\kappa^2 \omega^2 + \lambda^2 \zeta^2), \quad (9a)$$

$$4\pi \mathring{\tau}_m^{0j} = 4\pi \mathring{\tau}_m^{j0} = -(\kappa\omega \times \lambda\zeta)^j, \quad (9b)$$

$$4\pi \mathring{\tau}_m^{jk} = -(\kappa^2 \omega^j \omega^k + \lambda^2 \zeta^j \zeta^k) + \frac{1}{2}(\kappa^2 \omega^2 + \lambda^2 \zeta^2) \delta^{jk}, \quad (9c)$$

$$\mathring{\tau}_m^{\kappa\lambda} \equiv \eta \mathring{\tau}_m^{\circ\kappa\lambda}. \quad (9d)$$

The units of the constitutive parameters  $\kappa$  and  $\lambda$  are chosen to yield a stress-energy density. We usually choose  $\kappa=1$  as it is only the ratio of the barred to unbarred coefficients in Eq. (8) that are independent. The last line defines a stress-energy-dissipation tensor. The  $\omega$ -field is the generalization of the NST vorticity and the  $\zeta$ -field is the generalization of the swirl field (or Lamb vector) of the NST. Combining Eq. (9a) with Eq. (9d) gives the energy-dissipation rate of the fluid (Scofield and Huq, 2008). The other terms, using Eq. (9d), give the stress dissipation of the fluid (Scofield and Huq, 2009).

In Eq. (7a) the SAST Euler and NSEs are obtained by setting the second or third terms to zero, respectively. This removes the vortex field. Equation (7b) reformulates Eq. (7a) by replacing the vortex field stress-energy  $\mathring{\tau}_m^{\kappa\nu}$  by an equivalent Lorentz force  $-\mathring{F}^{\mu\nu} j_\nu$ . Thus, the vortex field is a source of disturbance to a SAST Navier–Stokes flow. For small flow velocities, this term persists, acting as a perturbation to the NSEs. The disturbances can be decomposed into transverse wave modes. These vortex waves may already have been observed in straight pipe (Faisst and Eckhardt, 2003; Hof *et al.*, 2004) and shear flow experiments (Waleffe, 1998). They have been analyzed in the context of the NSEs (Wedin and Kerswell, 2004) [see Eckhardt (2008), Eckhardt *et al.* (2007), and Kerswell (2005) for recent reviews]. They have also been observed in flows having curved geometries such as those found in helical pipes where they are known as Dean vortices (Dean, 1928; White, 1929; and Taylor, 1929). These have also been analyzed in the context of the NSEs (Wang, 1981; Yanase *et al.*, 1989; and Yamamoto *et al.*, 1998).

The evaluation of the vortex field contribution on the right-hand side of the first equation of Eq. (7b) requires new constitutive relations given in Eq. (7c) for which  $\mathring{H} = \star \mathring{F}$  is a simple example. Here,  $\star$  denotes the Hodge-star operator (Flanders, 1989, p. 15). In terms of these fields and the exterior derivative, the vortex field equations (7c) and (7d) are  $dH = 4\pi J$  and  $dF = 0$ . To complete the set of equations, either an equation of state for the pressure or the continuity equation (for an incompressible fluid) must be supplied. In the formulation of Eqs. (7), the special acoustic relativistic Euler equation for a perfect fluid is  $\tau_{e;\nu}^{\kappa\nu} = 0$ . The SAST generalization of the NSEs is  $(\tau_{e;\nu}^{\kappa\nu} - \tau_n^{\kappa\nu})_{;\nu} = 0$ .



An important point is that there is a vector potential  $A$  for the vortex field. This potential is self-consistently determined by the balance of fluid stresses whose components  $A_\mu$  are obtained by solving the wave equation (7e). This equation provides the potential for determining the propagation of transverse waves, not longitudinal sound waves. In that equation, the current  $j_\mu$  is the physical 4-momentum density. We have scaled  $j_\mu$  by the coupling parameter  $\bar{\eta}$  so that the equations have homogeneous units. All equations below the first two describe the vortex field and are called the *vortex field equations*. The wave equation for the vector potential components  $A_\mu$  can be solved by expanding  $A_\mu$  in a complete basis set of functions each of which satisfies the boundary conditions. This basis of transverse wave modes can also serve to compute the current 4-vector in Eq. (7e). These modes are the excitations of the flow. More complicated flows are expressed in terms of a series of such modes.

As previously noted, these equations are not the equations of a field of vorticity, as might be computed from the NST. The self-consistent field calculation procedure to determine their solutions amounts to sequentially solving the equations listed in Eqs. (7) (choosing either of the first two equations) with a starting approximation being a solution to the NSEs. Because the Lorentz force, appearing on the right-hand side of Eq. (7b), drives the evolution of what is essentially the acoustic space-time generalization of the NSEs, the question naturally arises: “what is the counterpart for viscous fluids having a vortex field corresponding to the helicity for inviscid fluids?”

### III. EVOLUTION OF HELICITY

Using GFD, we can exactly compute the scalar fluid helicity  $h=A \text{ curl } A$  in the form originally found by [Woltjer \(1958\)](#) for special types of vortex field configurations as given in Eq. (1). Actually, in this section we find not only a refinement but also a generalization of that notion of helicity. Furthermore, by relating this new notion of helicity to the Hopf invariant integral, we show that one can determine topological changes in a flow. We first present the derivation of a helicity evolution equation using vector analysis. Then, we generalize these results using the calculus of differential forms. This leads to an exact evolution equation for helicity evolution, Eq. (24) below and also provides an integral invariant formulation.

#### A. Vector-analytic derivation

We search for an equation that has the appearance of a conservation equation with the scalar helicity  $h_t=A \cdot \omega$  as the time-component of a 4-vector. If this is achieved with a nonvanishing source, then a helicity evolution equation will have been found. Using standard vector analysis, we therefore examine the time rate of change of  $h_t=A \cdot \omega$  given as follows:

$$\frac{\partial A \cdot \omega}{\partial t} = \frac{\partial A}{\partial t} \cdot \omega + A \cdot \frac{\partial \omega}{\partial t}. \quad (10)$$

For simplicity, we use units where  $c_m=1$ . Independent of Eq. (10), because  $dF=0$ , we have  $\nabla \cdot \omega=0$  and  $-\partial \omega / \partial t + \lambda \nabla \times \zeta=0$ . From  $\nabla \cdot \omega=0$ , we choose  $\omega=\nabla \times A$ . Because  $\nabla \times (\pm \nabla \phi)=0$  for any  $\phi$ , we then have

$$\nabla \times \left( \frac{\partial A}{\partial t} + \lambda \zeta + \nabla \phi \right) = 0. \quad (11)$$

Note as well that  $-\partial \omega / \partial t + \lambda \nabla \times \zeta=0$  is invariant with respect to the addition or subtraction of the gradient of a scalar to  $\zeta$ . We identify  $\phi$  with the scalar part of the 4-vector potential  $A$ , i.e., as in Eq. (7e). Using Eq. (11) so that  $\partial \omega / \partial t = \partial(\nabla \times A) / \partial t = -\nabla \times (\lambda \zeta + \nabla \phi)$  in Eq. (10), we obtain

$$\frac{\partial A \cdot \omega}{\partial t} = -(\lambda \zeta + \nabla \phi) \cdot \omega - A \cdot \nabla \times (\lambda \zeta + \nabla \phi)$$

$$= -(\lambda \zeta + \nabla \phi) \cdot \omega - A \cdot \nabla \times (\lambda \zeta). \quad (12)$$

A conservationlike law can be obtained from this expression if the right-hand side can be written as a divergence. For this, we use the following vector identities:

$$\begin{aligned} \nabla \cdot (\phi \omega) &= \omega \cdot \nabla \phi + \phi \nabla \cdot \omega = \omega \cdot \nabla \phi, \\ \nabla \cdot (A \times \lambda \zeta) &= \lambda \zeta \cdot (\nabla \times A) - A \cdot (\nabla \times \lambda \zeta), \\ &= \lambda \zeta \cdot \omega - A \cdot (\nabla \times \lambda \zeta). \end{aligned} \quad (13)$$

Thus,

$$\begin{aligned} \nabla \phi \cdot \omega &= \nabla \cdot (\phi \omega), \\ -A \cdot (\nabla \times \lambda \zeta) &= \nabla \cdot (A \times \lambda \zeta) - \lambda \zeta \cdot \omega, \end{aligned} \quad (14)$$

allowing Eq. (12) to be written as

$$\frac{\partial A \cdot \omega}{\partial t} = -2\lambda \zeta \cdot \omega + \nabla \cdot (-\phi \omega + A \times \lambda \zeta). \quad (15)$$

This can be rewritten as a helicity evolution equation

$$\frac{\partial A \cdot \omega}{\partial t} + \nabla \cdot (\phi \omega - A \times \lambda \zeta) = -2\lambda \zeta \cdot \omega. \quad (16)$$

In the last equation, if we multiply  $\lambda \zeta \cdot \omega$  by the absolute viscosity  $\eta$  which has units of  $ET/L^3$  and use the fact that  $[\lambda]=[U^{-1}]$ ,  $[\omega]=[U/L]$ , and  $[\zeta]=[u \times \omega]=[U^2/L]$ , we find  $[\eta \lambda \zeta \cdot \omega]=[ET/L^3 \times U^{-1} \times U^2/L \times U/L]=[E/T/L^3]$ . That is,  $\eta \lambda \zeta \cdot \omega$  has units of energy-dissipation rate per unit volume. The parameter  $\lambda$  is one of the four (two independent) new viscosity parameters that are related to the existence of the vortex field. When  $\lambda$  vanishes, as shown below, there is no production of helicity. In fact,  $\lambda$ -dependent quantities in the energy-dissipation tensor are responsible for a substantial increase of energy dissipation in the transition to turbulent flow, as shown in [Scofield and Huq \(2009\)](#). Equation (16) is linear, so we can multiply by  $\eta$  throughout. Below we will relate the integral of  $\zeta \cdot \omega$  to topological change. Thus, the source of topological change can be expressed in terms involving a quantity with the units of energy-dissipation rate per unit volume. This relates topological change to the phenomenological viscosity parameter  $\lambda$  and the alignment of the vorticity  $\omega$  with the swirl  $\zeta$  vector. The quantity  $\eta \lambda \zeta \cdot \omega$  is thus a measure of how lossy or dissipative the fluid is with respect to the vortex field. (The quantity  $\zeta \cdot \omega$  vanishes identically in the NST and for perfect fluids.) This is significant because as we show next, this term is related to the presence of and changes in the vortex field.

## B. Differential form derivation

The previous results are important enough to motivate a generalization using the tools of the exterior calculus of differential forms ([Nakahara, 2003](#); [Flanders, 1989](#)). Let  $\omega$  and  $\zeta$  denote the vectors comprising the *vortex field* 2-form  $F$  and their *vector potential* 1-form  $A$ ,

$$A = \phi dt + A_x dx + A_y dy + A_z dz,$$

$$F = \omega_x dy \wedge dz + \omega_y dz \wedge dx + \omega_z dx \wedge dy + \lambda(\zeta_x dx + \zeta_y dy + \zeta_z dz) \wedge c_m dt. \quad (17)$$

We then form a 3-form called the *topological torsion* density ([Kiehn, 2007](#))



$$h = A \wedge dA = A \wedge F. \quad (18)$$

From the 1-form  $A$ , we can form the 2-form  $dA = \frac{1}{2}F_{\mu\nu}dx^\mu dx^\nu$ . In GFD the components  $F_{\mu\nu}$  are the components of the vortex field. These are fluid analogs to the electromagnetic field tensor. The integral of the topological torsion density

$$\mathcal{H}_G = \frac{1}{16\pi^2} \int_{M^3} A \wedge dA = \frac{1}{16\pi^2} \int_{M^3} A \wedge F \quad (19)$$

has been shown (Nakahara, 2003, p. 372) to be a form of a *Hopf invariant* of algebraic topology.

In more detail, let the integral in Eq. (19) be extended to be over a 3D boundary  $\partial M^4$  of a four-dimensional (4D) manifold. Then, by Stokes theorem  $\mathcal{H}_G = (\frac{1}{16\pi^2}) \int_{\partial M^4} A \wedge dA = (\frac{1}{16\pi^2}) \int_{M^4} dA \wedge dA$ . By De Rham's first theorem (Flanders, 1989, p. 68) the integral vanishes when the 3-form  $\alpha = A \wedge dA$  is closed ( $d\alpha = 0$ ), or exact ( $\alpha = d\beta$ ), or if the boundary  $\partial M^4$  has a particular simple form, e.g., sphere, torus. By De Rham's second theorem, if the 3D domain is chosen not to be a boundary,  $\partial M^4$ , and if  $\alpha$  is closed but not exact (e.g., because of holes in the manifold), then the integral has a finite value, called the period that is an integer. It is this closed, but not exact, integrand that is related to the presence of nontrivial knots defining manifolds.

The Hopf (topological) invariant is related to the invariant degree of knottedness or linking of vorticity lines which are everywhere tangent to  $\omega$  (Ren *et al.*, 2007). Since the vorticity is solenoidal,  $\nabla \cdot \omega = 0$ , a vorticity line either closes in on itself or extends to infinity similar to a magnetic field. To change this degree of knottedness, linking, etc., requires physical processes of relinking. The more general definition, given in Eq. (19), relates the solenoidal field  $A$  to the generalization of its curl,  $dA$ , in a SAST. The nonvanishing of  $A \wedge dA$  implies that the system is not in a static equilibrium (Kiehn, 2007). Changes in this quantity thus indicate that the topology of the fluid system has changed.

Using Eqs. (17) and (18) to evaluate  $h$  leads to

$$h = h_t dx \wedge dy \wedge dz - h_x dy \wedge dz \wedge c_m dt + h_y dz \wedge dx \wedge c_m dt - h_z dx \wedge dy \wedge c_m dt, \\ *h \equiv -h_t dt + h_i dx^i = -(A \cdot \omega, \lambda A \times \zeta - \phi \omega). \quad (20)$$

The last term in Eq. (20) defines the helicity vector  $*h$  as the Hodge- $\star$  dual of the helicity three-form  $h$  in a SAST. This is also called the topological torsion vector (Kiehn, 2007). It is seen that the scalar part  $h_t \equiv A \cdot \omega$  is the counterpart to the electromagnetic helicity density  $A \cdot B$  of Eq. (1). Combining Eqs. (18) and (20), it is seen that the vanishing of  $A$  or  $\lambda$  is equivalent to the vanishing of helicity.

Next we introduce a four-form density called the *topological parity*,  $k$ , defined as follows (Kiehn, 2007):

$$dh = d(A \wedge dA) = dA \wedge dA + (-1)^{\deg A} A \wedge d^2 A \\ = dA \wedge dA = F \wedge F = 2\lambda \omega \cdot \zeta \Omega_4 \\ \equiv k \Omega_4. \quad (21)$$

Here,  $\Omega_4$  is the differential 4-space volume  $\Omega_4 = c_m dt \wedge dx \wedge dy \wedge dz$ . We call both  $k$  and the 4-form  $k\Omega_4$  the topological parity, as the context will distinguish the two.

The quantity  $k = 2\lambda \omega \cdot \zeta$ , when nonvanishing, serves as a multiplier expanding or contracting a differential 4-volume  $\Omega_4$ . This is an indicator of an irreversible process of an open nonequilibrium system. For a transverse electromagnetic wave, the quantity corresponding to  $\omega \cdot \zeta$  is  $E \cdot B$  and this is conserved at a value of zero in lossless media. The situation  $E \cdot B > 0$  occurs in a dissipative medium. In the case of a perfect fluid or the NST,  $\zeta = u \times \omega$ , then, in fact, the box product  $\omega \cdot \zeta = \omega \cdot u \times \omega = 0$ , so  $dh = F \wedge F = 0$ . Such flows are (Frobenius) integrable and therefore reversible. By

definition, perfect fluids are not dissipative so there is no topological change. When the fluid vortex field is calculated from the vector potential using GFD in the general dissipative case,  $\omega$  and  $\zeta$  are not necessarily perpendicular. Therefore, topological change can occur because  $dh = k\Omega_4 = 2\lambda\omega \cdot \zeta\Omega_4 \neq 0$ . This involves dissipation (Scofield and Huq, 2009).

The expression for  $h$  given in Eq. (20) can be exteriorly differentiated yielding

$$\begin{aligned} dh &= - \left( \frac{\partial h_t}{\partial t} + \frac{\partial h_x}{\partial x} + \frac{\partial h_y}{\partial y} + \frac{\partial h_z}{\partial z} \right) dx \wedge dy \wedge dz \wedge dt \\ &= k\Omega_4, \end{aligned} \quad (22)$$

which upon using the results from Eq. (20) gives

$$\frac{\partial h_t}{\partial t} + \nabla \cdot h = -k, \quad (23)$$

$$\frac{\partial A \cdot \omega}{\partial t} + \nabla \cdot (\phi\omega - \lambda A \times \zeta) = -2\lambda\omega \cdot \zeta. \quad (24)$$

This is the same result (with  $c_m=1$ ) stated in the vector-analytic derivation [Eq. (16)]. The first equation is a first order partial differential equation (PDE) with constant coefficients, involving the components  $h_\mu$  of the helicity vector. Here,  $k$  acts as a source for changing the helicity vector. Equation (24) reveals that the solution to the equation is more complicated. To solve it one might express  $\omega$  and  $\zeta$  in terms of the vector potential components. This leads to a second order nonlinear PDE in  $A$ . However, there is already a (wave) equation for the components of the vector potential equation (7e). Thus, Eq. (23) is a constraint on the vector potential. Since the vortex field is closed ( $dF=0$ ) but only locally exact ( $F=dA$ ), the constraint equation (23) can have different solutions in different regions of the flow. This reflects the existence of singular points in the vector potential or nonconnectedness of the topology of its domain of definition. The nonvanishing of the topological parity  $k$  is thus a characteristic of viscous flow having a vortex field as indicated by the constitutive parameter  $\lambda$ .

The Hopf invariant integral  $\mathcal{H}_G$ , defined by using the helicity vector coefficients  $h_t$  expressed as a 3-form, is an integer. So the differential contribution to the integral, while continuously evolving, leads to a quantity that is quantized by the integers and sourced by the topological parity  $k$ . Each time a new mode of the transverse wave excitation is generated, this integral changes because the vector potential  $A$  must change. This implies that the quantization of the helicity  $\mathcal{H}_G$  is associated with the creation of these transverse waves.

In the transitional flow régime (from laminar to turbulent flow) with transverse waves present, one finds the following contribution to the energy-dissipation rate (Scofield and Huq, 2009):

$$-\frac{d\mathcal{E}}{dt} = 2\eta\lambda \int_{\Delta} \omega \cdot \zeta d^3x. \quad (25)$$

Comparing this expression to the definition of topological parity,  $k=2\lambda\omega \cdot \zeta$ , and Eq. (23), we see that there is an close relation between the dissipation in the flow and the different topological states produced in the flow

$$\frac{d\mathcal{E}}{dt} = \eta \int_{\Delta} k d^3x = \eta \int_{\Delta} \left( \frac{1}{c_m} \frac{\partial h_t}{\partial t} + \nabla \cdot h \right) d^3x. \quad (26)$$

The energy-dissipation rate on the left is then related to the production of helicity density vector  $h$  in the flow, as indicated by the term in parenthesis on the right. The dissipation rate in Eq. (25) depends on the transverse wave modes excited. For viscous flows, to maintain a given topology in

steady state flow with particular modes excited, then a certain energy-dissipation rate must be maintained.

We have just shown that the topological parity  $k = F \wedge F = dh$  acts as a differential source of nonvanishing helicity. Thus,  $k$  is a source of topological change, as measured by the Hopf integral of the (3-form made from the) helicity density  $h$  vector. Note that the potential  $A$  is not the same as the velocity field, so the present conservation law is not the same as that suggested by Moffatt (1969). Equation (24) is additionally significant because it shows in 4D acoustic space-time that both a scalar helicity density  $h_t = A \cdot \omega$  and a vector “helicity current”  $\vec{h} = c_m(\phi\omega - \lambda A \times \zeta)$ , respectively, are involved in the helicity generation in a fluid driven by the topological torsion  $k$ . These are the scalar and vector parts of  $A \wedge dA$ .

The evaluation of the Hopf invariant in Eq. (19) when Stokes’ theorem is applied uses a fixed boundary for  $M^3$ , denoted by  $\partial M^3$ . If time-variable boundaries are chosen, e.g., ones that twist part of the 3D boundary in time, there is the likelihood of helicity flux entering or exiting the spacelike surfaces of  $\partial M^3$  between  $t_0$  and  $t_1$ . Such moving boundary surfaces can be used to describe the stirring of a body of fluid. (This is sensible as well: stirring a cup of tea introduces helicity into the fluid in the cup!). In these cases, boundary variations in time are also a source of helicity, generating a helicity flux. The integral conservation law, the Hopf integral equation (19) is thus complementary to the differential conservation law [Eq. (23) or (24)].

We have therefore found three things: conserved quantities and sources of topological torsion  $k$ , fixed 3-surfaces,  $\partial M^4$ , over which the quantities are conserved, and the fact that moving surfaces can generate helicity. In a 4D SAST, since  $\nabla \cdot \omega = 0$ , within GFD we have  $\omega = \nabla \times A$ . (Note in the NST or a perfect fluid, one defines  $\omega = \nabla \times u$ .) Then the helicity scalar takes the value  $h_t = A \cdot \omega = A \cdot \nabla \times A$ . This is the form found by Woltjer (1958). The condition  $h_t = A \cdot \nabla \times A = 0$  is the condition for Frobenius integrability in 3D. So the vanishing of the scalar part of the helicity vector implies integrability. If the velocity field was parallel to the vector potential,  $A = au$ , then there would be obtained in this special case the scalar form of the helicity density  $h = au \cdot \omega$  suggested by Moffatt (1969). In each of these cases it should be emphasized that the helicity 4-vector has four components [see Eq. (20)] not just a single scalar time-component. In the case of GFD, the topological analysis that is typically applied to electromagnetic magnetic fields can now be applied to the analogous vortex fields in fluid dynamics.

### C. Comparison

Upon comparing the vector-analytic approach and the one using differential forms, we see that the differential form analysis delivers a global result that is independent of smooth maps of the fluid manifold and indifferent to the dimension of the fluid space-time. The differential form calculation also shows that helicity can be generated along space-time boundaries. This is not evident from the vector-analytical approach.

## IV. SUMMARY AND CONCLUSIONS

In this paper, we extend the definition of fluid helicity from one applicable to inviscid fluids to one applicable to viscous fluids. Therefore, we no longer have to rely on an electromagnetic field analogy or an inviscid fluid assumption to understand the production of topological changes in a viscous fluid. By using the GFD, a self-consistent vortex field having a 4-vector potential is included in the description of the fluid. The new, generalized, definition of viscous fluid helicity  $\mathcal{H}_G = (16\pi^2)^{-1} \int_{M^3} A \wedge dA$  uses this vector potential. The vector potential also enables the computation of the topology changing transverse wave excitation modes of the flow. A direct link, Eqs. (21) and (25),  $\mu$  is developed between the rate of topological change due to these excitations and energy-dissipation rate due to the vortex field. This involves the phenomenological viscosity parameter  $\eta$ , the vortex field constitutive parameter  $\lambda$ , and the differing values of the projection  $\zeta \cdot \omega$  of the vorticity  $\omega$  onto the swirl  $\zeta$  vector components of the vortex field for each transverse wave mode. This is expressed by Kiehn’s topological parity  $k = 2\lambda \zeta \cdot \omega$ , a quantity that identically vanishes in the Euler theory of perfect fluids and in the NST. Therefore, the vortex field is

fundamental to the creation of topological change and leads to energy-dissipation in fluid flow. The implication is that the full vortex field is needed to determine topological change in fluid flows and that this field cannot be obtained from the NSEs. The theory of helicity evolution developed here, based on the existence of a computable vector potential, provides an effective probe for evaluating topological change in fluid flows. The topological and dynamical results obtained using GFD recommend it for the study of transitional and turbulent flows where the presence of topology changing transverse wave modes of the vortex field is important.

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