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Concordances among electromagnetic, fluid dynamical, and gravitational field theories

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ABSTRACT

We demonstrate mathematical concordances among the theories of electrodynamics, fluid dynamics, and gravitation when the latter two are extended by including a differential geometric structure that we call a vortex field. Experimental data and theoretical arguments for considering such vortex fields in these theories are discussed.

1. Introduction

This Letter provides a new analysis of three classical field theories: electrodynamics, fluid dynamics, and gravitation theory. The objective is to show that a physically motivated extension of these theories based on current conservation leads to concordances between the three theories. We then apply the understanding developed from the concordances to the problem of representing the curvature of a classical spacetime (gravitation). The analysis focuses on a differential geometric structure that generalizes the electromagnetic field tensor $F_{\mu\nu}$. The generalized structure has components analogous to the electric and magnetic field vector components found in $F_{\mu\nu}$. We show that such a structure should be included in a field theory whenever there are conserved currents in a spacetime, i.e., whenever the currents satisfy a continuity equation. Electrodynamics is an example of a field theory with such a differential geometric structure — the electromagnetic field. This new dynamical interpretation of the continuity equation suggests analogs of the electromagnetic field could be fruitfully included in extensions of fluid and gravitation field theories as they are also constrained by a continuity equation.

We show how such extensions can be developed using the vortex field theorem (VFT) described here. The VFT states that whenever there are conserved currents in a 4D spacetime, there are differential geometric structures analogous to the electromagnetic field. We call these differential geometric structures *vortex field structures*. Conversely whenever there are vortex fields, there are concomitant conserved currents. Because the vortex field generates stress-energy, the vortex field must be included in the stress-energy flux balance for each theory [1]. This leads to similar equations for the vortex field, continuity and stress-energy flux balance.

The concordances are significant because results for one of the continua become transferable to others by scaling, not just by analogy. For instance, the present work using the VFT refines the analogies between electrodynamical theory (EDT) and fluid dynamics [2]. For the latter, we obtain an extension of the Navier–Stokes theory of fluids called geometro-fluid dynamics (GFD) with an integral fluid dynamical vortex field [3]. This is important because the geometrodynamical theory provides a vector potential from which the topological change of a continuum during its evolution can be examined [4–7]. This can form a basis for a fluid model of spacetime where the evolution of whorls, voids, etc., can be tracked.

The analogy between fluid flow and the evolution of a spacetime also can be used to study gravitational black hole analogs using longitudinal acoustic waves [8–10], see Ref. [11] for a recent review. As another example of using the VFT to extend a classical field theory, we develop in this Letter a perfect fluid model of an evolving spacetime that satisfies the VFT. We show that this locally becomes equivalent to general relativity theory (GRT) when the spacetime stress-energy tensor in the perfect fluid model is replaced by a metric approximation. We use the perfect fluid model without these local or any weak field approximations to show how the shrinking of an initially spherical galaxy produces a vortex and an axial jet similar to those observed in spiral galaxies.

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2. Background

The present development originates from consideration of the wave-like excitation modes of a physical continuum. We classify these excitation modes according to

(i) amplitude,
(ii) type: longitudinal or transverse with respect to their direction of propagation, and
(iii) degree of coupling to the ambient flow.

Longitudinal waves generally require the medium to be compressible. Small amplitude longitudinal sound waves carry little energy for nearly incompressible fluids such as water. Small amplitude, short wavelength, high frequency, transverse waves exist in fluids but are evanescent [12]. However, large wavelength transverse waves can and do persist to influence the fluid dynamics [13]. These transverse waves have been experimentally observed in helical flows [14, see esp. Fig. 2], [15]. For this flow field, the transverse waves are strongly coupled to the ambient fluid and are sustained by energy transfer from the ambient flow. This strong coupling modifies both the vortex field and the ambient flow field. Experimental support for the existence of such transverse wave modes of fluid dynamical vortex fields has been recently presented [3,13]. These transverse waves have a maximum speed of propagation c_m where $c_m \ll c$. That is, the dynamics of transverse waves in fluid flows is non-relativistic because the maximum speed of fluid transverse waves is very much lower than the speed of light c. The existence of this speed limit c_m is most easily enforced by introducing a special acoustic spacetime. These considerations as well as corroborating experimental evidence resulted in the formulation of the geometro-fluid dynamics (GFD) which includes a fluid vortex field [3,13]. This theory has the mathematical structure of a relativistic fluid theory with a speed of propagation c_m replacing c and includes a fluid vortex field. This leads one to consider the inclusion of the vortex field in other theories of continua and to study their resulting similarities, i.e., to consider concordances.

2.1. Vortex field theorem

To develop the mathematical concordances for the three field theories, we state and prove the vortex field theorem (VFT) linking vortex fields to conserved currents in a spacetime. Preliminary to this we introduce some definitions. Namely, for a 4D manifold, given an arbitrary 4-vector with contravariant components $(J^\mu) = (J^t, J^x, J^y, J^z)$, the relation ($c_m = 1$)

$$\frac{\partial J^t}{\partial t} + \frac{\partial J^x}{\partial x} + \frac{\partial J^y}{\partial y} + \frac{\partial J^z}{\partial z} = 0 \tag{1}$$

is called the *J-conservation condition* or continuity equation. For our purposes J is a 4-vector current. We use methods from differential geometry for the following calculations [16,17]. To derive a continuity equation, we associate a (differential geometric) density 3-form ${}^\star J$ in the Minkowski spacetime with the 4-vector current. This involves using the components of the vector $(J^\mu) = (J^t, J^x, J^y, J^z)$ and the four-volume $\Omega_4 = dt \wedge dx \wedge dy \wedge dz$. The 3-form *density* ${}^\star J$ is then defined by the interior product of Ω_4, denoted $i_J(\Omega_4)$, along the direction of the vector J as [18]

$$\begin{aligned} {}^\star J \equiv i_J(\Omega_4) = &+ J^t\, dx \wedge dy \wedge dz - J^x\, dt \wedge dy \wedge dz \\ &+ J^y\, dt \wedge dx \wedge dz - J^z\, dt \wedge dx \wedge dy. \end{aligned} \tag{2}$$

We use a metric $(\eta_{\mu\nu}) = \mathrm{diag}(-1, 1, 1, 1)$ for the Minkowski metric tensor and include its effects on the Hodge-$\star$ of differential forms in a spacetime [16, p. 46]. Raising/lowering of the time-component indices changes their sign. The exterior derivative of the current density ${}^\star J$ can be evaluated as follows

$$\begin{aligned} d\,{}^\star J &= d\big(J^t\, dx \wedge dy \wedge dz - J^x\, dy \wedge dz \wedge dt \\ &\quad + J^y\, dx \wedge dz \wedge dt - J^z\, dx \wedge dy \wedge dt\big) \\ &= d\big(J^t\, dx \wedge dy \wedge dz\big) + d\big(-J^x\, dy \wedge dz \wedge dt \\ &\quad + J^y\, dz \wedge dt \wedge dx - J^z\, dt \wedge dx \wedge dy\big) \\ &= d\big(J^t\, dx\big) \wedge dy \wedge dz + (-1)^1 J^t\, dx \wedge d(dy \wedge dz) \\ &\quad + d\big(-J^x\, dy \wedge dz \wedge dt + J^y\, dz \wedge dt \wedge dx - J^z\, dt \wedge dx \wedge dy\big) \\ &= \left(\frac{\partial J^t}{\partial t} + \frac{\partial J^x}{\partial x} + \frac{\partial J^y}{\partial y} + \frac{\partial J^z}{\partial z}\right) dt \wedge dx \wedge dy \wedge dz. \end{aligned} \tag{3}$$

If the quantity in parentheses in the last line above vanishes, then the *continuity equation*, Eq. (1), is satisfied. Since the differential 4-form volume Ω_4 multiplying it never vanishes along any evolutionary trajectory defined by J, the continuity equation implies the conservation of current density $d\,{}^\star J = 0$. We call the condition $d\,{}^\star J = 0$ the *${}^\star J$-conservation* condition. Conversely from Eq. (3), $d\,{}^\star J = 0$ implies the continuity equation. Using the equivalence of *${}^\star J$-conservation* and a continuity equation leads to the formulation of the vortex field theorem.

Table 1
Comparison of $(3+1)$D vortex field theory (VFT) to electromagnetic theory (EMT).

VFT	EMT
$\kappa \nabla \cdot \omega = 0$	$\nabla \cdot B = 0$
$\frac{\kappa}{c_m} \frac{\partial \omega}{\partial t} + \lambda \nabla \times \zeta = 0$	$\frac{1}{c} \frac{\partial B}{\partial t} + \nabla \times E = 0$
$\bar{\lambda} \nabla \cdot \xi = 4\pi \rho_\xi$	$\nabla \cdot D = 4\pi \rho_D$
$-\frac{\bar{\lambda}}{c_m} \frac{\partial \xi}{\partial t} + \bar{\kappa} \nabla \times \varpi = \frac{4\pi}{\bar{\eta}} J_\xi$	$-\frac{1}{c} \frac{\partial D}{\partial t} + \nabla \times H = 4\pi J_D$

Theorem 1 *(Vortex Field Theorem (VFT)). For a simply connected* 4D *spacetime manifold with conserved currents, ${}^\star J$-conservation* (*equivalently, the continuity equation*) *implies the vortex field equations* (*Table* 1) *and conversely in a spacetime, a vortex field has a conserved current density.*

The proof of the VFT depends on the converse of the Poincaré lemma (CPL [16, p. 27]). According to the CPL, if we have a conserved 3-form current density ${}^\star J$, i.e., $d\,{}^\star J = 0$, then there is a 2-form such that $dH = 4\pi\, {}^\star J$. By Poincaré's lemma $d(H + F) = dH + dF = dH + d^2 A = dH$, whenever $F = dA = \frac{1}{2}(A_{\nu,\mu} - A_{\mu;\nu})\, dx^\mu \wedge dx^\nu$, where F is a 2-form and A is a 1-form. The gauge potential A serves to define F and the latter provides a gauge transformation of H. If we have $dH = 4\pi\, {}^\star J$, then because $d^2 H = 0 = 4\pi\, d\,{}^\star J$, we have current density conservation. To construct a physical theory, the 2-form H can be related to the 2-form F by a constitutive equation in terms of their components: $H_{\kappa\lambda} = C^{\mu\nu}_{\kappa\lambda} F_{\mu\nu}$. For the case of a vacuum spacetime one can use the Hodge-$\star$ to express the constitutive relations as $-{}^\star H = F$. When these components are expressed in the simplest manner for a spacetime, we obtain the equations in Table 1.

The vortex field theorem shows that the existence of current conservation is equivalent to an analog of Maxwell's equations. While this is a simple result, it is not trivial: its significance seems to have been gone unrecognized in fluid mechanics, except as an analogy. It has also gone unrecognized in electromagnetic field theory that the continuity equation implies Maxwell's field equations, although the converse is easily demonstrated: operate with ∂_t on $\nabla \cdot D = 4\pi\rho$, evaluate $\nabla \cdot (\partial_{ct} D - \nabla \times H = -4\pi J)$ and combine to obtain the continuity equation. In both of these theories the conservation of current is often adopted as an auxiliary condition.

For the case of electromagnetic theory the importance of the vortex field equations is clear: We see that the mathematical form of electromagnetic field theory can be derived from the continuity equation. For the case of fluid dynamics, the vortex field must exist for the same reason as in electromagnetic theory: a conserved current. The VFT then implies that vortex field effects, experimentally observed in helical pipe flow [13], should exist generally as excitations in other flows.

The vortex field can be decomposed in terms of transverse wave modes as mentioned in the background section. In Table 1 we also compare the vortex field equations to Maxwell's equations for the transverse wave fields in electromagnetic field theory. The similarity of these equations shows the concordance among field theories having a vortex field and the classical electromagnetic field theory. Here $\{\lambda, \bar{\lambda}, \kappa, \bar{\kappa}, \bar{\eta}, c_m\}$ are material constitutive parameters that can be determined from experiment. The parameter $\bar{\eta}$ provides the coupling of the current to the vector potential. The vortex field equations imply the existence of a vector potential A whose components A_μ satisfy a wave equation

$$-\Box A_\mu \equiv -\left(\partial^2_{c_m t} - \nabla^2\right)A_\mu = 4\pi J_\mu. \tag{4}$$

The vortex field components $F_{\mu\nu} = (A_{\nu,\mu} - A_{\mu,\nu})$ determined by the vector potential A involve the vorticity ω and swirl ζ fields, a nomenclature derived from fluid dynamics. These quantities are given in Eq. (5) below. The wave equation describes transverse waves in which a component of the field is orthogonal to the direction of propagation of the wave.

3. Three classical continuum theories

The physical basis of the unified approach developed for the three field theories studied in this paper is the existence of transverse waves and conserved currents in the continuum. The mathematical basis is the vortex field theorem (VFT). The presence of a vortex field modifies the stress-energy flux balance in the continuum. Below we give a comparison of the vortex field theory (also abbreviated VFT) to electromagnetic field theory (EMT). Then, by including the stress-energy of a matter field, mathematically consistent with the vortex field, we are able to compare the extended electrodynamical theory (EDT), fluid dynamics (GFD), and gravitation (GFT). The GFT leads to Einstein's view of gravitation via an additional step called *kinematization* described below whereby the spacetime stress-energy in the dynamic stress-energy flux balance is replaced using a quasi-static, local metric approximation. We also consider relevant experimental or observational corroboration of the theories.

3.1. Comparison of VFT to EMT

The VFT equations can be cast into the form of Maxwell's equations in a $(3+1)$D representation as shown in Table 1. Mixed Dirichlet–Neumann boundary conditions can be used: specifying values for ζ and normal derivatives of ω on boundaries. In Table 1 c_m is the maximum speed of transverse waves, c is the speed of light, and $\bar{\eta}$ gives the coupling of the current to the vortex field. This coupling parameter does not appear in Maxwell's equations. The VFT equations are analogous to the Maxwell field equations and they provide a new basis for the derivation of the Maxwell field equations for a 4D continuum having conserved currents. Both EMT and VFT define a wave equation (re. Eq. (4)) from which their respective vortex fields $F_{\mu\nu}$ can be computed. Because of the linear dependence of the vortex field F on the vector potential A, the modes of the vortex field F will also satisfy a wave equation.

Table 2
Stress-energy flux balance in three theories. Here $j^\nu = (\rho_q c_m, \rho_q u^i)^\nu$. Conserved currents j^ν are coupled to a vortex field as described in Table 1.

EDT	$(\tau_e + \tau_m)^{\mu\nu}{}_{;\nu} = 0,$	$(\tau_e)^{\mu\nu}{}_{;\nu} = F^{\mu\nu} j_\nu$
GFD	$(\tau_e - \tau_n - \tau_m)^{\mu\nu}{}_{;\nu} = 0,$	$(\tau_e)^{\mu\nu}{}_{;\nu} = (\tau_n)^{\mu\nu}{}_{;\nu} - F^{\mu\nu} j_\nu$
GFT	$(\tau_e - \tau_{st})^{\mu\nu}{}_{;\nu} = 0,$	$(\tau_e)^{\mu\nu}{}_{;\nu} = -F^{\mu\nu} j_\nu$

3.2. Stress-energy flux balances

The current or velocity field must be specified to completely determine the dynamics of a physical medium. Classical field theories specify these quantities using stress-energy or stress-energy flux balances. These equations must be formulated in a way that is compatible with the symmetry of the vortex field. This locally requires Lorentz covariance in the tangent space of the underlying manifold using the maximum speed of the transverse waves c_m. More generally, for electrodynamical theory (EDT), geometro-fluid dynamics (GFD), and the perfect fluid spacetime gravitational field theory (GFT), the stress-energies must be tensors. As seen in left hand column of Table 2, the stress-energy-flux balance equations have the same general form. The right-hand column gives the same results as the left, but uses the identity $\tau^{\mu\nu}{}_{;\nu} = -F^{\mu\nu} j_\nu$ for both the vortex field stress-energy τ_m and the spacetime stress-energy τ_{st}. The semicolon indicates covariant differentiation.

In Table 2, $\tau_e^{\mu\nu} = \rho u^\mu u^\nu + p(\eta^{\mu\nu} + c_m^{-2} u^\mu u^\nu)$ is the Lorentz transformation covariant stress-energy of the mass distribution. Here p is the pressure and ρ the density. The stress-energy $4\pi\tau_m^{\mu\nu} = \eta^{\mu\alpha} F_{\alpha\beta} F^{\beta\nu} - \frac{1}{4}\eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}$ is a symmetric, traceless matrix. The tensor τ_{st} has the same mathematical form as τ_m. The Newtonian fluid stress energy for a fluid with absolute viscosity η is given again by a symmetric matrix having components $\tau^n_{\mu\nu} = 2\eta(\tilde{\sigma}_{\mu\nu} + \delta\theta\mathcal{P}_{\mu\nu})$. In the latter, the stress-energy projected into a spacelike 3-volume is given by $\tilde{\sigma}_{\mu\nu} = \frac{1}{2}(u_{\mu;\epsilon}\mathcal{P}^\epsilon_\nu + u_{\nu;\epsilon}\mathcal{P}^\epsilon_\mu) - \frac{1}{3}\theta\mathcal{P}_{\mu\nu}$ where the projection operator is given by $\mathcal{P}_{\mu\nu} = \eta_{\mu\nu} + u_\mu u_\nu$, $(\eta_{\mu\nu}) = \mathrm{diag}(-1, 1, 1, 1)$ is the spacetime metric tensor, and $\theta = u^\mu_{;\mu}$. The 4-velocities (u_μ) satisfy the normalization condition $(c_m = 1)$ $u^\mu u_\mu = -1$ [1,4,19].

The equations in the right-hand column of Table 2 can be interpreted as mass distribution fluxes driven by a with stress-energy flux originated by various force fields. Each of the three field theories, involve a Lorentz force or analog $(\tau_m)^{\mu\nu}{}_{;\nu} = -F^{\mu\nu} j_\nu$ that is evaluated as follows:

$$\left(F^{\mu\nu} j_\nu\right) = \begin{pmatrix} 0 & \lambda\zeta_1 & \lambda\zeta_2 & \lambda\zeta_3 \\ -\lambda\zeta_1 & 0 & \kappa\omega_3 & -\kappa\omega_2 \\ -\lambda\zeta_2 & -\kappa\omega_3 & 0 & \kappa\omega_1 \\ -\lambda\zeta_3 & \kappa\omega_2 & -\kappa\omega_1 & 0 \end{pmatrix} \begin{pmatrix} -j_0 \\ j_1 \\ j_2 \\ j_3 \end{pmatrix}$$
$$= \begin{pmatrix} +\lambda\zeta_1 j_1 + \lambda\zeta_2 j_2 + \lambda\zeta_3 j_3 \\ \lambda\zeta_1 j_0 + \kappa\omega_3 j_2 - \kappa\omega_2 j_3 \\ \lambda\zeta_2 j_0 - \kappa\omega_3 j_1 + \kappa\omega_1 j_3 \\ \lambda\zeta_3 j_0 + \kappa\omega_2 j_1 - \kappa\omega_1 j_2 \end{pmatrix}, \tag{5}$$

so that

$$F^{i\nu} j_\nu = \left(\lambda\zeta^i j_0 + \kappa(j\times\omega)^i\right), \quad i = 1, 2, 3,$$
$$F^{0\nu} j_\nu = \lambda\zeta^i j_i. \tag{6}$$

The ζ^i and ω^i terms constitute the vortex field. We have used the metric $(\eta_{\mu\nu}) = \mathrm{diag}(-1, 1, 1, 1)$ to lower the indices of the current in Eq. (5). The first line of Eq. (6) describes how, for instance, the electromagnetic vortex field Lorentz force accelerates the mass distribution. The term $F^{0\nu} j_\nu = \lambda\zeta^i j_i$ in the last line represents the rate at which the swirl field does work on the mass density. In Eq. (6) the term $-(\lambda\zeta^i j_i)$ is the rate at which the mass distribution looses energy to the vortex field. The term $-(\lambda\zeta^i j_0 + \kappa(j\times\omega)^i)$ is like a (negative) Lorentz force per unit volume acting on the mass-distribution to slow it down.

The equations in Table 2 are similar. The differences in signs appearing in the Lorentz force analogs in the Table 2 depend on the field system description, leading to concordances rather than a unification of the three field theories. For the first case, electrodynamics (row EDT), the equations are formulated for a thermodynamically closed, conservative system: Overall, the rate of change of mass stress-energy equals the force of the electromagnetic vortex field acting on it. In particular, the rate of change of the mass-energy equals the rate at which the electromagnetic vortex field does work on the mass distribution $(\tau_e)^{0\nu}{}_{;\nu} = F^{0\nu} j_\nu$ and the rate of change of momentum of the mass equals the electromagnetic vortex field Lorentz force acting on it $(\tau_e)^{i\nu}{}_{;\nu} = F^{i\nu} j_\nu$.

For GFD, second row, the system is a thermodynamically open, non-conservative, the Newtonian viscous stress-energy contributes to the energy dissipation rate and the balance of stress-energy flux. This means that the fluid vortex field instead of doing work on the fluid to accelerate the mass distribution extracts work and slows it down. To keep the flow going requires, e.g., a pressure drop as energy is dissipated instead of being preserved as in the conservative system of electrodynamics. The vortex field structure is seen to make an additional contribution to the local stress-energy flux balance. The formulation for the fluid gravitation theory (third row GFT) is formulated in the same way as the fluid case, omitting the Newtonian viscous stresses. The motion of matter leads to the radiation of a gravitational vortex field.

3.3. *Electrodynamics*

Electrodynamics involves the coupling between the electromagnetic (vortex) field and the matter field. Separately, the energy–momentum of the matter field and the vortex field are conserved. When the two are coupled, their sum satisfies a flux balance equation. The first equation for the electrodynamical field theory (EDT) case given in Table 2 describes this balance of stress-energy flux. The equations $(\tau_e + \tau_m)^{\mu\nu}{}_{;\nu} = 0$ describe a conservative system: the rate of change of the inertial stress-energy τ_e is balanced by an opposite rate of change in the stress-energy τ_m. The structure of the stress-energy τ_m described above allows this balance to be written as the right-most column in the EDT row, since $(\tau_m)^{\mu\nu}{}_{;\nu} = -F^{\mu\nu} j_\nu$. This relation, derived from the vortex field $F^{\mu\nu}$ and current conservation, is common to the other two cases cited in Table 2. Electrodynamics is the paradigm example of a vortex field coupled theory satisfying the VFT and has extensive experimental verification.

3.4. *Geometro-fluid dynamics*

The second pair of equations in Table 2 are the GFD stress-energy flux balance equations. These equations have the same form as those for electrodynamics (EDT) except for the sign of the vortex field contribution τ_m and the contribution due to the Newtonian viscous stress τ_n. The sign difference occurs because EDT is a conservative system where an increase of energy in the electromagnetic field occurs at the expense of a decrease of energy in the matter field: a moving charge radiates an electromagnetic field and thereby slows down. In GFD the effect of the vortex field is to drive the flow. This can be described in terms of an excitation of a base flow by transverse wave modes of the vortex field. In GFD when the vortex field stress-energy vanishes, $\tau_m = 0$, the system reduces to the *form* of the relativistic Navier–Stokes equations for a maximum speed of transverse waves of $c_m \ll c$ [19, Ch. XV]. Both the Newtonian fluid viscous stresses τ_n and the vortex field derived stress τ_m drive the motion of the mass distribution: $(\tau_e)^{\mu\nu}{}_{;\nu} = (\tau_n + \tau_m)^{\mu\nu}{}_{;\nu}$. Therefore by including the effects of a vortex field, the GFD equations extend the Navier–Stokes equations [3]. GFD limits to the Navier–Stokes equations when $c_m \to \infty$ and the vortex field vanishes. Because the sign of the fluid dynamical Lorentz force effects are negative compared to electrodynamics, the mass-distribution loses energy–momentum when it excites the vortex field.

Experimental observations of the vortex field, i.e., transverse wave modes have been shown to explain the substantially increased energy dissipation rates observed in helical flows (compared to the Navier–Stokes theory) [3,13]. This augmented energy dissipation has a linear and a quadratic power law dependence on the velocity field that is successfully explained in GFD by the vortex field stress-energy dissipation rate tensor $\eta\tau_m$. (Here η is the absolute viscosity [3].) In the helical flow experiments, a turbulent flow is from a straight section to a helical section where it relaminarizes into a main flow plus a vortex field flow consisting of transverse wave modes. On exit, the vortex field modes persists for thousands of diameters downstream after exit into another straight section [15]. This fluid vortex field is supported by the analog of an electrodynamical Lorentz force given by $(\pm\tau_m)^{\mu\nu}{}_{;\nu} = \mp F^{\mu\nu} J_\nu$ appearing on the right-hand side of the equations in the right column of Table 2. As the pressure drop across the helical flow guide is further increased the number of vortex field modes increases until a turbulent state is observed [15].

The GFD equations include the Newtonian viscous fluid stress τ_n. In the limiting case of vanishing vortex field the GFD equation limit to the Navier–Stokes equations. For a perfect fluid, the Newtonian viscous stresses vanish. Here one has to consider steady vortex fields analogous to steady magnetic and swirl fields as well as time-dependent excitations. In the case of perfect fluids all three cases become nearly identical in form. The GFD (as a field theory) is the second example of a vortex field theory satisfying the VFT.

3.5. *Gravitational theories*

Many theories of spacetime invoke a perfect fluid model [4,20]. For such models of spacetime, the VFT implies that there should be a spacetime fluid vortex field. This has implications with respect to the expected rotation of galaxies since the vortex field would be an additional source of forces acting on the stars, gas, and spacetime in the galaxy [21]. Therefore, as a third example of vortex field coupled systems, we consider the implications of the vortex field theorem for such theories of spacetime. The additional spacetime stresses could be interpreted as anomalous gravitational forces additional to Newtonian gravitation. The gravitational vortex field in this case represents an energy–momentum loss mechanism due to gravitational radiation.

Of particular interest is the transition from a flat spacetime theory of gravitational forces into a curved metric theory of gravitation where forces are replaced by a curved spacetime such as in general relativity theory (GRT). This transition is required of all theories that "geometrize" starting with a physical basis for the underlying spacetime, e.g., a perfect fluid spacetime. Here we use an incompressible perfect fluid model for the flat spacetime preceding the geometrization step. A Bose–Einstein condensate (BEC) or superfluid cold dark matter is an example of this scenario [21, 22]. The spacetime currents in such perfect fluids satisfy a continuity equation and are therefore subject to the VFT. We call the model a gravitational field theory (GFT) because the interactions in the spacetime are all gravitational with like masses attracting.

This model gives the third set of equations in Table 2. Compared to GFD, the viscous dissipation term τ_n has been omitted. The spacetime stress-energy tensor τ_{st} has the same mathematical form as τ_m, being derived from the vortex field $F^{\mu\nu}$ of spacetime. In, for instance a perfect fluid model, the stress-energy flux of matter equals that of the spacetime. This model becomes a general relativity theory (GRT) when a mass distribution (generalized to a mass 4-current) curves a *kinematical spacetime* defined by the fluid flow. By this we mean that it is possible to replace the spacetime stress-energy by the metric geometry of a curved spacetime. This allows a separation of the physics of the underlying model and the curved spacetime aspects of the gravitation problem.

In order to obtain GRT, we kinematize GFT as follows: For a local, quasi-static or steady-state balance of stress-energy, an increase in mass stress-energy is accompanied by an increase in spacetime stress-energy. So the local stress-energies can be taken to satisfy a simpler relation than in Table 2 for GFT, namely the constraint

$$\tau_e^{\mu\nu} - \tau_{st}^{\mu\nu} \equiv 0. \tag{7}$$

(We use the sign convention of Ref. [1], see Ref. [4] for a list of other choices.) This relation satisfies the stress-energy flux condition of the GFT in Table 2 and clearly includes the spacetime stress-energy $\tau_{st}^{\mu\nu}$. Eq. (7) reflects a static view of spacetime where the stress-energy of spacetime is exactly that created by matter. Any material parameters required for balancing this equation are assumed present in the definition of mass and spacetime stress-energy. The spacetime is not rapidly evolving in this approximation. We then apply the Cartan–Weyl approximation theorem for symmetric tensors. This theorem states that *locally* the most general approximation of a symmetric tensor (to first order in the second derivatives of a metric tensor $g^{\mu\nu}$) is given by [23,24]

$$\tau_{st}^{\mu\nu} \approx \beta\sqrt{-g}\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right) + \lambda\sqrt{-g}g^{\mu\nu}. \tag{8}$$

Here $R_{\mu\nu} = R^{\kappa}_{\mu\nu\kappa}$, $R = R^{\kappa}_{\kappa}$ where $R^{\kappa}_{\mu\nu\epsilon}$ is the Riemannian (torsion free) curvature tensor and β and λ are constants. This metric defines a *kinematic geometry* arising from the spacetime energy–momentum density $\tau_{st}^{\mu\nu}$. Because the energy–momentum tensor of the vortex field spacetime is symmetric, the spacetime is pseudo-Riemannian with symmetric metric tensor $g^{\mu\nu}$ [1]. At this point the explicit stress-energy of the gravitational field has disappeared. By using the Cartan–Weyl approximation, it is clear that the Einstein tensor arises from the stress-energy of the spacetime that is coupled to the mass $\tau_e^{\mu\nu}$. So in a sense, this term contains the gravitational effects in the way originally envisioned by Einstein, namely that mass leads to the curvature of a kinematic spacetime. We then obtain using Eq. (8) and the definition $G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R$ (also known as the Einstein tensor):

$$G^{\mu\nu} = \kappa\tau_e^{\mu\nu} + \Lambda g^{\mu\nu} \tag{9}$$

as a *local steady-state approximation*. Relative to Eq. (8), we have set $\kappa = \beta^{-1}/\sqrt{-g} = 8\pi G_N/c^4$ and $\Lambda = -\lambda/\beta$. Eq. (9) becomes Einstein's gravitational field equation when $\kappa = 8\pi G_N/c^4$ where c is the speed of light and G_N is Newton's gravitational parameter. It states a relationship between the deviatoric Ricci tensor $G_{\mu\nu}$ and the stress-energy tensor of matter $\tau_e^{\mu\nu}$. For a finite speed of transverse waves c_m, locally the metric $g^{\mu\nu}$ limits to the Minkowski metric in the tangent space as implied by the VFT. The standard approach to evaluate the phenomenological parameters in Eq. (9) uses an asymptotic limit as the region of applicability of the equation is enlarged without limit (Birkhoff's theorem [25, p. 253]). This traditional procedure thus omits the large-scale spacetime currents that can persist in this asymptotic limit. On the other hand, the fact that the GFT limits to GRT in this limit suggests that GFT is applicable to the problems solved by GRT as well as being applicable to more strongly coupled evolution rate dependent and long range slow field problems not described by GRT.

The result given in Eq. (9) does not, at first glance, appear to contain the stress-energy $\tau_{st}^{\mu\nu}$ of the spacetime gravitational field [4]. However, the stress-energy of the gravitational field has merely been hidden by the Cartan–Weyl approximation, Eq. (8). The result embodied in Eqs. (7)–(9) thus provides a clarification of the gravitational energy paradox in GRT whereby the gravitational stress-energy of spacetime does not explicitly appear. (See Ref. [20, p. 70], for instance.) The Cartan–Weyl approximation hides the vortex field structure generating $\tau_{st}^{\mu\nu}$ since this quantity is replaced in the kinematization procedure.

Because of the success of the local approximation Eq. (9), where $\kappa = 8\pi G_N/c^4$, it is seen that the gravitational kinematic vortex field has transverse wave modes that locally travel at the speed of light waves. We therefore set $c_m \to c$, in the GFT although the maximum speed of spacetime transverse waves (gravitational waves) might be lower. For large distances, there can appear anomalous spacetime currents that are not explained by the local approximation of GFT leading to GRT. We can account for the forces caused by these currents and surmount this difficulty by using the full GFT.

3.6. Vortex field of a perfect fluid model galaxy

To verify that the vortex field extension of gravitation theory may be useful in the same way as Newtonian gravitation theory or GRT, we show that there exists a global spacetime gravitational potential for the spacetime manifold M. To further verify the importance of the VFT based theory, we then provide an illustrative example calculation of the vortex field strength $F_{\mu\nu}$ of a manifold consisting of an evolving polarizable spherical fluid spacetime inclusion in an otherwise quiescent perfect fluid spacetime environment. This allows us to present an explicit expression of $F_{\mu\nu}$ for a cosmologically significant problem. The quasi-static local metric approximation and the Cartan–Weyl approximation theorem can be applied at any point to make contact with GRT but this is not needed in the present analysis especially since we will be considering the large scale limit where the vortex field is important compared to a flat spacetime.

That a global potential exists follows as a corollary of the Chern–Weil theorem [16, p. 149], [17, Ch. 11]. Here the trace curvature, is a global invariant polynomial with respect to coordinate transformations over the base spacetime manifold M, independent of the local moving frame. By the theorem it is then closed ($dF = 0$) and exact ($F = dA$). Furthermore, for a classical vacuum spacetime, we can employ the Hodge-$\star$ and the VFT to determine the conserved spacetime currents from $d{}^\star F = 4\pi\, {}^\star J$.

Let us now examine the structure of this trace curvature on a base spacetime manifold M in terms of the Riemann curvature 2-form matrix [4, p. 340]:

$$R^{\mu}_{\nu} = \frac{1}{2}\left(\frac{\partial\Gamma^{\mu}_{\nu\lambda}}{\partial x^{\kappa}} - \frac{\partial\Gamma^{\mu}_{\nu\kappa}}{\partial x^{\lambda}} + \Gamma^{\rho}_{\nu\lambda}\Gamma^{\mu}_{\rho\kappa} - \Gamma^{\rho}_{\nu\kappa}\Gamma^{\mu}_{\rho\lambda}\right)dx^{\kappa}\wedge dx^{\lambda}. \tag{10}$$

By setting $\mu = \nu = n$ and summing, the 2-form trace Riemann curvature indicated by the Chern–Weil theorem is found to satisfy

$$R^{n}_{n\kappa\lambda}\,dx^{\kappa}\wedge dx^{\lambda} \equiv F_{\kappa\lambda}\,dx^{\kappa}\wedge dx^{\lambda} = \left(\frac{\partial\Gamma^{n}_{n\lambda}}{\partial x^{\kappa}} - \frac{\partial\Gamma^{n}_{n\kappa}}{\partial x^{\lambda}}\right)dx^{\kappa}\wedge dx^{\lambda}.$$

Since the trace curvature 2-form F has a potential, $F = dA$, we have the non-trivial result that

$$F_{\mu\nu} = \frac{\partial A_{\nu}}{\partial x^{\mu}} - \frac{\partial A_{\mu}}{\partial x^{\nu}} = \frac{\partial\Gamma^{n}_{n\nu}}{\partial x^{\mu}} - \frac{\partial\Gamma^{n}_{n\mu}}{\partial x^{\nu}}. \tag{11}$$

This shows that without approximations there is a 1-form $A = A_{\lambda}\,dx^{\lambda}$ such that $F = dA$ which is related to the connection coefficients $\Gamma^{\kappa}_{\mu\nu}$ modulo a gauge transformation $A \to A + d\chi$. By the VFT there are conserved spacetime currents defined by $d{}^\star F = 4\pi\, {}^\star J$. Note that taking the trace has linearized the theory with respect to the connection. The result of this abelianization is a $\mathfrak{u}(1)$ gauge theory such as that developed in the VFT. That is, the gauge field strength of the curvature $F_{\mu\nu}$ is a vortex field.

The vortex field $F_{\mu\nu}$ that follows from the Chern–Weil theorem and the VFT is not quite the Ricci tensor in a weak-field approximation as the contraction of the Riemann curvature tensor in the present and the weak-field cases involves different indices. In fact in Eq. (10) one cannot contract on indices to obtain the Ricci tensor. When one applies the Cartan–Weyl approximation theorem, Eq. (8), as a first step thereby replacing the spacetime vortex field energy–momentum tensor $\tau_{st}^{\mu\nu}$ by an approximation, one obtains the standard results of GRT as described in Ref. [20, Sections 4.3–4.4]. The latter results are obtained by using tensor analytic methods involving $R^{\mu}_{\nu\kappa\lambda}$, and contracting on μ and κ. The weak-field linearization of the equations of GRT then leads to an electromagnetic analog including spin-2 metrical gravitational waves [20, pp. 74–76]. In contrast the present approach gives an EMT analog without approximations.

To find the full kinematical spacetime caused by the vortex field energy–momentum $\tau_{st}^{\mu\nu}$ and a mass distribution, one solves the field equations in Table 2 for $\tau_{st}^{\mu\nu}$, then uses Eq. (8) in the last step to find the metric $g^{\mu\nu}$ of the kinematic spacetime:

$$\begin{aligned} 4\pi\tau_{st}^{\mu\nu} &= g^{\mu\alpha}F_{\alpha\beta}F^{\beta\nu} - \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \\ &\approx \beta\sqrt{-g}\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right) + \lambda\sqrt{-g}g^{\mu\nu}, \end{aligned} \tag{12}$$

thereby locally describing a kinematic spacetime including the effects of the vortex field in a way that is consistent with GRT. This approach is not limited by the use of the Cartan–Weyl approximation theorem. The latter is merely used here to construct a local, quasi-static, kinematic spacetime. The trace curvature of the Chern–Weil theorem always has a stress-energy tensor of the form given in the first line of Eq. (12). Thus we can always apply the Cartan–Weyl theorem locally as a quasi-static approximation and generate a GRT-like spacetime. This spacetime might not have a simple global structure as we find for our example developed below. We note that the symmetric metric has 10 parameters and the vortex field has 6. The determination of all these parameters must be made for a complete geometrical description of the spacetime. When this is done, the present theory describes a curved spacetime and an electromagnetic analog (vortex field) that is not limited to a weak-field or small scale, quasi-static, approximation. These results imply there are spin-1 vortex field gravitational effects and spin-2 weak-field metric gravitational effects. The low speed limit to the equations of motion for a perfect fluid yield Euler equations in which the effect of the vorticity and swirl are included well below any turbulent behavior [3]. This indicates that such vortical motion should be commonplace in a universe if it can be well represented as a perfect fluid.

We now apply these results to model galactic swirls and jets. By assuming that the material parameters $\{\lambda, \bar{\lambda}, \kappa, \bar{\kappa}, \bar{\eta}, c_m\}$ are not wave-vector and frequency dependent, our model, but not the over-all theory, is restricted to the slow motions and large scales of galaxies. We determine a field strength 2-form $F = \mathrm{tr}(\frac{1}{2}R^{\varepsilon}_{\kappa\mu\lambda}\,dx^{\mu} \wedge dx^{\nu}) \equiv \frac{1}{2}F_{\mu\nu}\,d\xi^{\mu} \wedge d\xi^{\nu}$ that can serve as an abelianized gauge potential. Starting with Cartesian coordinates (x, y, z, w) on an S^3 galaxy of radius $a[\eta]$, evolution variable η, we transform to three (χ, θ, ϕ) coordinates, then define basis one-forms

$$\begin{aligned} &\xi^{\hat{\chi}} = a[\eta]\,d\chi, \qquad \xi^{\hat{\theta}} = a[\eta]\sin\chi\,d\theta, \\ &\xi^{\hat{\phi}} = a[\eta]\sin\chi\sin\theta\,d\phi, \qquad \xi^{\hat{\eta}} = d\eta, \end{aligned} \tag{13}$$

where the radius $a[\eta]$ varies with the epoch parameter η. (Not to be confused with the coupling parameter $\bar{\eta}$ above.) The orthogonal moving frame is then calculated. This allows the differential connection $\Gamma = (\Gamma_i^j[dx^{\lambda}])$ and the field strength 2-form or field $F = d\Gamma + \Gamma \wedge \Gamma$ of the shrinking galaxy to be computed [4, p. 357] $(\ell, k = \hat{\chi}, \hat{\theta}, \hat{\phi})$

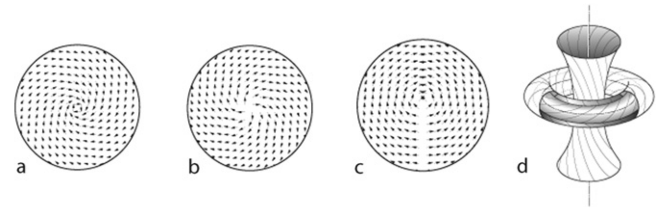

Fig. 1. Direction field of the anomalous gravitation field ω of a galaxy for $\ddot{a}/a = -1$ in a conventional right-hand Cartesian coordinate system with z upwards. (a), (b) (x, y)-plane disks $z = (0.45, 0.0)$, (c) (z, x)-plane disk, $y = 0.0$, (d) schematic of particle path lines. There is a small constant, inward directed field ζ also due to the conserved spacetime currents leading to accelerated galactic shrinking [26–32].

$$\left({}^{(3)}F^k_{\ell}\right) = \left(\frac{1+\dot{a}^2}{a^2}\xi^k \wedge \xi^{\ell}\right), \qquad F^{\hat{\eta}}_k = \frac{\ddot{a}}{a}\xi^{\hat{\eta}} \wedge \xi^k. \tag{14}$$

The field strength 2-forms given in Eq. (14) is the abelianized curvature of the galactic spacetime which according to Eq. (5) gives the ζ and ω fields. We are interested in the excitation or ${}^{\star}F$ field strength figuring in $d{}^{\star}F = 4\pi\,{}^{\star}J$. Galactic evolution is modelled by specifying the rate of change of the scaled galactic radius with respect to the evolution parameter η, $\dot{a}[\eta] = -c_r \cdot \sin[\eta]$. The evolution parameter has the same units as a so that $\dot{a}$ is unitless. The rate parameter c_r is empirical – it depends on the dynamics of the shrinkage, a topic beyond the present work. The spatial part of the curvature is homogeneous and isotropic for all values of the evolution parameter η. The spatial–temporal part is similarly homogeneous and isotropic. In the transformation of the coordinates (χ, θ, ϕ) to (x, y, z), it is found that the values for the vortex field strength are double valued for the full range $0 \leqslant \chi < \pi$ as the other two angular variables cover their ranges. Summing forces from both $0 \leqslant \chi < \pi/2$ *and* $\pi/2 \leqslant \chi < \pi$ sheets by identifying points with the same (x, y, z)-values preserves the local geometry but changes the global topology to one homeomorphic to the 3D projective plane $\mathbb{RP}^3$. This gives for the vortex field excitation ${}^{\star}F$ with components ${}^{\star}F_{\mu\nu}$ satisfying $d{}^{\star}F = 4\pi\,{}^{\star}J$. (See Eqs. (5)–(6).)

$$\begin{aligned} &\omega_1 = -\frac{\ddot{a}(yr - zx)}{ar\rho}, \qquad \zeta_1 = -\frac{x}{r}\left(\frac{1+\dot{a}^2}{a^2}\right), \\ &\omega_2 = +\frac{\ddot{a}(xr + zy)}{ar\rho}, \qquad \zeta_2 = -\frac{y}{r}\left(\frac{1+\dot{a}^2}{a^2}\right), \\ &\omega_3 = -\frac{\ddot{a}\rho^2}{ar\rho}, \qquad \zeta_3 = -\frac{z}{r}\left(\frac{1+\dot{a}^2}{a^2}\right). \end{aligned} \tag{15}$$

Here $\rho = (x^2 + y^2)^{1/2}$. The ω-field jet defining a left-handed spacetime flow direction-field has been oriented so that the jet is upward and plotted in Fig. 1a, b, c for selected slices through the galaxy. Fig. 1d provides a schematic of the path lines of particles in the galaxy. It is seen that the evolution of the S^3 ($\rightarrow \mathbb{RP}^3$) manifold has broken the (x, y, z)-spherical symmetry. The resulting (ζ, ω)-field constitutes an analog to electromagnetic radiation whose emission diminishes the size of the galaxy. This "radiation" represents a transverse massive spacetime mode that is strongly coupled to the ambient spacetime. Thus, its speed of propagation may be substantially less than the speed of light, even though the maximum speed would be still be c. The ζ-field given in Eq. (15) consists of a (presumably small) spherically symmetric *inward* directed constant "Pioneer effect" force [26–28]. This inward directed flux is the consequence of the shrinking of the S^3 ($\rightarrow \mathbb{RP}^3$) manifold. This effect produces a negative density core cusp mollifying the positive core cusp found in CDM simulations, e.g., [29,30]. These results do not include "normal gravitational mass" effects due to black holes, stars, or free hydrogen gas which form the observable galactic disk – a mere 10% effect. Just *kinematic* effects are modelled. These additional effects require the solution of the field equations including the stress-energy of normal matter as given in Table 2. Only at this point needs one invoke the hypothesis of a perfect fluid. From our calculation it is seen that the structure of galaxies, modulo the 10% normal matter effect, is governed by a simple dynamics [31]. Given detailed measurements of spacetime currents, the constitutive parameters in Eq. (5) can be found and the universality of this dynamics tested.

4. Summary

This Letter describes a mathematical concordance among *extensions* of three classical field theories: electrodynamical theory (EDT), geometro-fluid dynamics, (GFD) and develops a perfect fluid model of spacetime that leads to a gravitational field theory (GFT) in which spacetime currents are used to explain the anomalous vortex structure and jets of spiral galaxies. Experimental and observational support for these extensions of fluid dynamical and gravitational theories are discussed. The extensions result from the application of the vortex field theorem (VFT). The VFT shows that whenever there are conserved currents, there will be vortex fields – analogs to an electromagnetic field. Thus it is necessary to include the vortex field in the stress-energy flux balance equations for a continuum. This leads to all three theories having a mathematical concordance due to a vortex field. The differences in these vortex field continuum theories then consists of the kind of currents satisfying the continuity equation, the coupling of the currents to the matter fields, and the details of the stress-energy flux balances used to determine the conserved currents.

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