

Fluid dynamical Lorentz force law and Poynting theorem—introduction

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Abstract

Fluid dynamical analogs of the electrodynamic Lorentz force law and Poynting theorem are introduced. Analysis shows the fluid dynamical Lorentz force and Poynting theorem describe new channels of stress-energy propagation and dissipation due to the action of longitudinal and transverse modes of flow. These channels are important for describing stress-energy dissipation along curved stream tubes such as those found in turbulent flows.

1. Introduction

This paper provides a *heuristic development* of the fluid dynamical analogs of the Lorentz force law and Poynting theorem of electrodynamics. A companion paper [1] rigorously derives these fluid dynamical analogs from the recently developed geometrodynamical theory of fluids (GTF, [2–5]). The new theory is a covariant, causal 4D spacetime theory of fluid flow. In the paper we explore the physical implications of GTF and compare it to the Navier-Stokes theory (NST). GTF contains equations isomorphic to the equations of electrodynamics. Thus the mathematical structure of the GTF equations can be used to derive the Lorentz force law and Poynting theorem for fluids [1]. This allows identification of new avenues of stress-energy propagation and dissipation in fluid flow beyond those contained in the NST. Our results are expected to have significant impact on the theoretical foundations of fluid dynamics and its applications including theories of vortex dynamics, astrophysical structure, and turbulent flow.

The motivation of the present work arises from experimental measurements of the stress-energy in helical flows or curved stream tubes of turbulent flows. This dissipation can be significantly larger than predicted by the NST. For example, the experimentally measured energy dissipation in helical flows can exceed the NST predictions by more than 300% [6].

Describing this dissipation clearly requires new pathways or avenues of energy-dissipation that are not contained in the NST. Our focus on the Lorentz force and Poynting theorem arises from the fact that Lorentz forces and the Poynting theorem energy and momentum transport can all be measured. Thus the theory presented can be experimentally tested.

The paper is organized as follows. The background section reviews some of the experimental support for the present theory including experimental measurements and observations of the isomorphism of parts of electrodynamics and fluid dynamics. We then discuss theoretical limitations of the NST formulation, especially the consequences of the diffusive character of the NSEs. The limitations of the NST account of stress-energy dissipation is then examined. The fluid dynamical analogs of the Lorentz force and the Poynting theorem of electrodynamics insightfully reveal new avenues of stress-energy dissipation that account for the experimental measurements of enhanced energy dissipation. There follows a section describing the integration of the Lorentz force into equations for fluid flow and its relation to the NST. A summary and conclusion section follows.

2. Background

In the NST, the standard method to account for the large values of energy-dissipation in turbulent flows is to introduce eddy viscosities [7]. This procedure preserves the form of the NST by replacing the newtonian viscosity with an eddy viscosity which can be many orders of magnitude larger than the newtonian viscosity of the fluid [7]. The limitations of such approximations are well known [8]. The introduction of this phenomenology to turbulence theory has a difficult theoretical justification [9, 10]. A physical theory with such large corrections is not only unaesthetic, it suggests new physics is needed [11–13].

Because of these and related problems, stress-energy dissipation and stress-energy propagation has been analyzed using electro-dynamical analogies of various kinds; in particular in [14–16] and more recently by Kambe [17]. The work of Kambe is noteworthy for its development of a more thorough electro-dynamical analogy and for its use of the gauge theoretic aspects of such analogies [18].

The fluid dynamical theory used here, the GTF, is distinguished from these theories in that it is a covariant spacetime theory. This means it is form invariant with respect to transformations of the spacetime. The transformations of most interest for spacetimes are ones which leave the transverse wave speed constant [19]. In the GTF this speed is denoted c_m . This speed is a material property and so is expected to vary from fluid to fluid. The GTF equations contain a subset of equations that have the same mathematical form as Maxwell's equations. Thus, these equations are said to be isomorphic to Maxwell's equations; both theories can describe transverse waves. In comparing these theories $c_m \ll c$ where c is the speed of light. This isomorphism is the basis for the analogy used in this paper to explain the meaning of GTF.

Both GTF and electromagnetic field theory (EMT) are *causal theories* meaning causes can be related to effects. This requires a finite speed of propagation so that disturbances at a given point come from distant points in a sequential order, thereby allowing cause and effects to be related. *Acausal* theories differ by being characterized by action-at-a distance or infinite speeds of propagation. For acausal theories, disturbances at a given point coming from even infinite distances arrive *simultaneously* with those from less distant points, complicating the task of relating causes and effects. The equations of EMT are invariant with respect to their form under Lorentz transformations leaving the speed of light c a finite constant. The fluid flow equations of GTF share this form invariance for a speed c_m . We call this invariance c_m -

Lorentz covariance. As we describe later in more detail, the Navier-Stokes equations fall into the action-at-a-distance category and are not c_m -Lorentz covariant. A formulation of the Navier-Stokes that is Lorentz covariant applicable to fluid flow has been developed by Eckhart [20] and Landau and Lifshitz [21]. However, their theory suffers from the action-at-a-distance problem inherent in the NSEs. (See the end of section 2.2.2.) This naturally motivates the development of a causal, spacetime formulation of fluid dynamics.

2.1. Experimental

In this section we explain the physics underlying the GTF using an electrodynamic-fluid dynamical analogy. Experimental observations provide striking confirmation of the value of this analogy for interpreting the structure of fluid flows. For this interpretation, we note much of electrodynamics is described in terms of the propagation of transverse (wave) modes through wave guides generated by solutions of Maxwell's equations. Thus, it can be anticipated that transverse modes also would be found in guided fluid flows. This is supported by experimental observations and theoretical calculations of modal structure in helical flow guides (Dean vortices [22, 23]), Taylor-Green flows [24], Taylor-Görtler flows [25], counter-rotating vortex pairs downstream of a jet in a cross-flow [26], toroidal liquid flow in stirred tanks [27, 28], as well as excitations of toroidal flows in air [29] can be interpreted as the generation and propagation of flows with transverse mode excitations. These experimental results, give clear evidence for the existence of the transverse modes in fluid flow.

In electrodynamics one finds three different kinds of these transverse modes, namely (i) transverse magnetic (TM) guided modes, (ii) transverse electric (TE) guided modes, and (iii) transverse electromagnetic modes (TEM) for propagation in unbounded geometries [30]. The various different modes are denoted by subscripts, e.g., TE_{mn} or TM_{mn} depending on the mode type. (TE having the electric field transverse to propagation direction or TM having the the magnetic field perpendicular to the direction of propagation.) In fluid dynamics, we call the analogous modes transverse vorticity (TV), transverse swirl (TS) and transverse vorticity-swirl (TVS) modes, respectively [2]. The GTF vorticity field ω is the analog of the magnetic field B and the GTF swirl field ζ is the analog of the electric field E . These relationships are developed more fully in the remainder of the paper.

To complete this section, we describe one of the clearest experimental examples of steady transverse mode fluid flow, that of flow through a helical flow guide of circular cross-section subsequently feeding into a straight flow guide section of the same circular cross-section [31, 32]. A turbulent flow enters the helical flow guide, then 'laminarizes' into a flow conventionally described as Hagen-Poiseuille flow with an embedded pair of counter-rotating vortices propagating downstream. Since helical flow consists of simultaneous rotation and translation about an axis transverse to the flow, these observations suggest the helical geometry selects a transverse mode flow. The most often observed mode is a counter-rotating vortex pair called a Dean flow [23]. (See [4] for a more detailed description.) The Dean vortex pair is a transverse vorticity, TV_{11} , mode. The two vortices are not independent; the motion of one vortex 'influences' the other one and conversely: the pair constitutes one mode. This dual vortex mode is the analog of the transverse magnetic, TM_{11} , mode of an electromagnetic field in a circular cross-section, helical electromagnetic wave guide. Experimental measurements show the fluid TV_{11} mode persisting for hundreds of diameters downstream after exit from the helical structure [32]. Other modes have been observed, for example, a transverse vorticity TV_{21} mode has been measured in curved airflow around a bend [33]. The 'excitation' of these flow modes can be qualitatively explained within newtonian physics as being due to the action of Coriolis forces. Such forces mimic the part of the Lorentz force $\rho \mathbf{u} \times \boldsymbol{\omega}$ generated

by the fluid analog to the electromagnetic field in curved stream tubes. (This can be seen by using a local curvilinear coordinate system consisting of a path variable ds , a radius r , and an angular variable θ . Since $ds = r d\theta$, the Coriolis force is proportional to $\rho u \times d\theta/ds$.)

2.2. Theoretical

In this section we present the background theoretical analysis leading from the Navier-Stokes equations to the geometrodynamical theory of fluids (GTF) including a more formal treatment of the electro-dynamical-fluid dynamical isomorphism. We first examine the propagation speed of small disturbances or signals in the fluid from a theoretical standpoint. A clear distinction emerges between theories requiring finite versus infinite speeds of propagation; this allows one to distinguish between causal and acausal theories. We next examine the stress and vorticity evolution provided by the NST and that theory's limitations with respect to its newtonian physics formulation. For expository purposes, we then introduce the electromagnetic and fluid dynamic Lorentz force law and Poynting theorems as electromagnetic analogies. We name the fluid dynamical analog of the electromagnetic field the fluid dynamical vortex field or *vortex field* for short. The vortex field equations are introduced below [2]. We show the fluid vortex field-electromagnetic field analogy can be derived as an isomorphism between their theories. Integral to the development of the theory is the distinction of galilean and Minkowski spacetimes. This allows us to replace the action-at-a-distance (galilean spacetime) formulation of the Navier-Stokes equations (with its action-at-a-distance and attendant infinite speeds of propagation) by a physically robust set of causal fluid flow equations having finite speeds for the propagation of flow perturbations (Minkowski spacetime).

2.2.1. Finite maximum speed and causality. Of fundamental concern in understanding the dynamics of fluids is the kinds of flow excitations and their speed of propagation. This is important since elementary considerations show the excitations that fluid elements can exhibit may be considered to consist of a superposition of rigid body translation-rotation as well as compressional, and torsional modes [21]. Lower frequency longitudinal and transverse (shear) waves lead to net motion of the fluid. The existence of a maximum phase velocity c_m profoundly constrains these modes [2]. In the following we discuss experimental observations of such upper velocity limits for compressional and volume-preserving torsional modes. We also relate these finite speeds to the causal propagation of disturbances.

Let us consider longitudinal sound waves first. These modes are described in the continuum approximation by a second order (scalar) wave equation. Infinite speeds of compressional (longitudinal) sound are predicted by the NST theory of incompressible fluid flow. This infinite speed propagation of sound (action-at-a-distance) for incompressible fluids is completely consistent with the galilean spacetime of newtonian physics for which the Navier-Stokes equations (NSEs) are formulated. When a finite compressibility is allowed, the (longitudinal) mode of sound propagation is described by a second order wave equation for the scalar density fluctuations with a finite speed of propagation. This finite speed removes this source of acausality. The speed of longitudinal mode propagation depends on material properties of the fluid. In most theories this is the compressibility and the shear modulus [34, section 2.2]. The non-physical effect of an infinite speed of sound might be ignored, since real fluids are compressible and longitudinal sound waves are often weak. However, the fundamental problem is deeper as indicated above.

The next kind of sound waves we need to consider are volume-preserving transverse ones [19]. These modes are described in the continuum approximation by a second order (vector)

wave equation. For such transverse flow modes, fluid lamella are in relative shear motion. Normally, as the shear modulus is small, these modes are weak [34]. In the NST, the resistance to shear leading to kinetic energy dissipation is called a viscous force. These transverse modes have finite speeds of propagation that depend on material properties [35–37]. The experimental measurements of propagation speeds range up to the order of $3 \times 10^3 \text{ m s}^{-1}$. These speeds are, of course, much less than the speed of light $c = 2.9979 \times 10^8 \text{ m s}^{-1}$, so the usual ‘relativistic effects’ for speeds near c are not of concern. The measurements of [29] show a finite speed of transverse mode propagation. The speed is also found to depend on wave number. This dependence is also a prediction of GTF. For White’s experiments on helical flow [22], we deduce an effective phase speed of transverse waves of about 150 m s^{-1} [4].

The mathematical expression of the problem of action-at-a-distance formulation of the Navier-Stokes equations involves noting they contain first order time derivatives of the velocity but second order spatial derivatives of the velocity arising from the viscous stress term. As we discuss in detail in the next section, this implies their classification as parabolic partial differential equations: consequently their unequal treatment of space and time contravenes the requirements of causality [38]. Thus, the problem with the NSEs is very deep. The weak causality violation, which we term *acausality*, has a source in the fact that solutions of diffusion equations, such as the vorticity diffusion equation (See section 2.2.2 below.) derived from the NSEs, have as elementary solutions a gaussian distribution function that is nowhere vanishing. This means for an infinitesimal time interval after the introduction of a point disturbance, the effects of the disturbance are found everywhere as the tail of the gaussian never vanishes. We conclude physical reasonableness (causality) requires a new consideration of the dynamics of fluid flow, especially the newtonian viscous term.

These considerations also can be understood in terms of the differences between galilean spacetimes and Minkowski spacetimes. Galilean spacetime is used for newtonian physical theories such as the Navier-Stokes equations. Galilean spacetime theories support (instantaneous) action-at-a-distance and are therefore termed acausal as discussed above. In cases where there is no time dependence, the action-at-a-distance paradigm makes sense. In this case equations such as the Navier-Stokes equations become elliptic partial differential equations. Here, there is no question of acausality as there is no ordering of events. If this time invariance symmetry is broken, then questions of causality re-emerge. This has important consequences for the range of applicability of the NSEs. In fact the NSEs give quite good results for time-independent flows where the equations reduce to an elliptic boundary value problem formulation [39].

In summary, we see the first step to developing a causal theory is to employ a finite speed of propagation of disturbances c_m . The mathematical apparatus required is that of the special theory of relativity, albeit for a much smaller propagation speed than that of light. This allows one to replace the galilean spacetime by the Minkowski spacetime, where causes and effects can be related by arrival sequence. The rigorous formulation of the equations of motion embodying these concepts is presented in the companion paper [1]. Here we proceed along a more intuitive path.

2.2.2. Vorticity, stress-energy dissipation, and stress flux balance according to the NST. Let us now summarize the Navier-Stokes theory of vorticity, stress-energy dissipation and stress generation in a fluid. In doing so, we note some theoretical shortcomings of the NST. We also use this discussion as a vehicle to transition from the standard vector component notation to a tensor notation in anticipation of the tensor notation used in the remainder of this paper and its companion [1]. In the Navier-Stokes theory, $\nu = \eta/\rho$ is the kinematic viscosity with η being

the absolute or ‘molecular’ viscosity; ρ is the fluid density. The fluid pressure is denoted by p ; velocity components by u^i . The equations describing the evolution of vorticity are commonly developed by converting the Navier-Stokes equations (NSEs) into an equation for the NST vorticity $\Omega \equiv \nabla \times u$. One starts with the NSEs for the velocity components u^i in vector component notation:

$$\frac{Du_i}{Dt} = \frac{\partial u^i}{\partial t} + u \cdot \nabla u^i = -\frac{1}{\rho} \nabla^i p + \nu \nabla^2 u^i, \quad i = 1, 2, 3, \quad (1)$$

takes the curl across the equation, assuming an incompressible fluid, so $\nabla \cdot u = 0$, then uses the definitions

$$\Omega \equiv \nabla \times u, \quad \Lambda \equiv (\nabla \times u) \times u = \Omega \times u. \quad (2)$$

to obtain the Navier-Stokes vorticity diffusion equations [5, 40],

$$\frac{\partial \Omega}{\partial t} + \nabla \times (u \cdot \nabla u) = \nu \nabla^2 \Omega \implies \frac{\partial \Omega}{\partial t} - [\Omega, u] = \nu \nabla^2 \Omega. \quad (3)$$

The identity $\nabla \times \nabla \Phi = 0$ is used to eliminate the curl of the pressure gradient $\nabla \times \nabla p$. The identity $u \times (\nabla \times u) = \nabla \frac{u^2}{2} - u \cdot \nabla u$ and the definition of the Lie derivative of Ω along u , given by

$$\mathcal{L}_\Omega u \equiv [\Omega, u] = \Omega \cdot \nabla u - u \cdot \nabla \Omega, \quad (4)$$

were used to obtain the last of equation (3). Equation (3) describes the evolution of NST vorticity Ω . The left-hand side of the equation defines the time varying Lie or ‘fisherman’s’ derivative $\partial_t \Omega - [\Omega, u]$ of the field Ω along the velocity field u . For an inviscid fluid or high Reynolds numbers, the right-hand side equation (3) vanishes, so the equation then describes the diffusion of the conserved NST vorticity Ω in the flow of a perfect fluid. For viscous fluids, the vorticity diffusion equation is first order in time and second order in the spatial derivatives. Mathematically, since equation (3) does not contain the same order (i.e., second) derivatives in all space and time coordinates, it is a parabolic equation describing the diffusion of vorticity in a galilean spacetime allowing (infinite speed) action-at-a-distance [38, section 3.3]. According to equation (3), the Navier-Stokes vorticity Ω evolves over all space at each (galilean) time-slice. Because of its diffusive nature, the Navier-Stokes vorticity diffusion equation can neither describe the finite speed propagation of transverse wave modes nor the propagation of stress-energy associated with those modes [2].

The energy-dissipation described by the Navier-Stokes theory is kinetic energy loss derived from the newtonian stress term in the Navier-Stokes equations. This dissipation is given by the Lamb-Stokes theorem [5, 41, p.581],

$$\frac{d\mathcal{E}_{KE}}{dt} = \frac{D}{Dt} \int_M \frac{1}{2} \rho u^2 dV = \frac{D}{Dt} \int_M e_{KE} dV = -\eta \int_M \|\Omega\|^2 dV - \eta \int_{\partial M} \Lambda \cdot dS, \quad (5)$$

giving the time rate of loss of kinetic energy \mathcal{E}_{KE} in terms of a viscosity η , the volume integral of the Navier-Stokes vorticity Ω and the Navier-Stokes swirl $\Lambda = \Omega \times u$. The last term on the right-hand side is a surface integral whose value depends on the boundary conditions. The kinetic energy density of the fluid is given above as $\frac{1}{2} \rho u^2$. The integrands in equation (5) are entirely point functions of local position and time—they are local fields whose values change as a function of time. As discussed above, these field values have effects with diffused amplitudes arriving at infinite speed to the field point from distant points. Using the same methods used to derive equation (5) we find the following diffusion equation for the kinetic energy density e_{KE} in the absence of body forces:

$$\frac{De_{KE}}{Dt} = \frac{\eta}{\rho} \nabla^2 e_{KE} - \nabla \cdot \left(\frac{p}{\rho} u \right) - \frac{\eta}{\rho} \|\nabla u\|^2. \quad (6)$$

The combination of the first term on the left and right-hand side describe the convective *diffusion* of kinetic energy. Such stress-energy *diffusion* is quite different from the GTF predictions of stress-energy *propagation* found below using the fluid dynamical Poynting theorem.

For completeness and to further introduce the tensor notation used in the paper, let us develop the NSEs with the assumption of a symmetric viscous stress using cartesian tensor notation. In this context, the newtonian viscous stress used in the NST is defined by $\tau_{ij} = -p\delta_{ij} + 2\eta\tilde{\sigma}_{ij}$. If this ansatz is substituted into the Cauchy equation of motion (stress flux balance equation [21]),

$$\rho \frac{Du^i}{Dt} = \frac{\partial \tau^{ij}}{\partial x^j} + \rho g^i = \tau_{,j}^i + \rho g^i, \quad (7)$$

we obtain the Navier-Stokes equations [39]. The Cauchy equation of motion is derived in terms of the balance of forces in the fluid. The Einstein summation convention is used: repeated tensor indices are summed over. The comma-subscript-index denotes partial differentiation with respect to the coordinate with the indicated index. The stress tensor (τ^{ij}) has the matrix representation

$$\left(\tau^{ij} \right) = \begin{pmatrix} \tau^{11} & \tau^{12} & \tau^{13} \\ \tau^{21} & \tau^{22} & \tau^{23} \\ \tau^{31} & \tau^{32} & \tau^{33} \end{pmatrix}. \quad (8)$$

Substituting equation (8) into equation (7), the Cauchy equations of motion can be written as

$$\begin{aligned} \rho \frac{Du^x}{Dt} &= \frac{\partial \tau^{xx}}{\partial x} + \frac{\partial \tau^{xy}}{\partial y} + \frac{\partial \tau^{xz}}{\partial z} + \rho g^x, \\ \rho \frac{Du^y}{Dt} &= \frac{\partial \tau^{yx}}{\partial x} + \frac{\partial \tau^{yy}}{\partial y} + \frac{\partial \tau^{yz}}{\partial z} + \rho g^y, \\ \rho \frac{Du^z}{Dt} &= \frac{\partial \tau^{zx}}{\partial x} + \frac{\partial \tau^{zy}}{\partial y} + \frac{\partial \tau^{zz}}{\partial z} + \rho g^z. \end{aligned} \quad (9)$$

For cartesian tensor analysis there is no difference between covariant components (τ_{ij}) and contravariant components (τ^{ij}). This is generally not the case for curvilinear coordinate systems. The covariant derivatives in cartesian coordinate systems are ordinary partial derivatives denoted here by a comma [42, 43].

In order to generalize the equations to arbitrary coordinate systems the partial differentials can be replaced by covariant derivatives leading to the equation [21]

$$\rho \frac{Du^i}{Dt} = \tau_{;j}^i + \rho g^i, \quad \tau^{ij} = -p\delta^{ij} + 2\eta\tilde{\sigma}^{ij}. \quad (10)$$

The only notational difference in the first of these equations and equation (7) above is the introduction of the semicolon on the right-hand side of the first equation indicating a covariant derivative. The left-hand side of equation (7) includes part of the effects of the fluid inertia. The equations describe a balance between the inertia flux and the newtonian stress flux due to the symmetric part of the velocity gradients and the partial derivatives of the pressure. The term ρg^i represents the introduction of a body force to the problem. The use of the spatial

covariant derivatives in equation (10) renders the form of the equation covariant with respect to spatial coordinate system changes. However, the equations are not covariant with respect to transformations between coordinate systems in relative motion if transverse mode propagation at finite speed is included in the theory. The problem arises from the implicit use of a galilean space time in which the time variable is completely unconnected from the spatial coordinates.

2.2.3. Continuity Equation. The continuity equation is an axiom of both the NST and the GTF. The continuity equation is usually solved along with the NSEs for incompressible flow problems so that the pressure field can be determined. The conservation of currents has been shown to imply the field equations of GTF, the vortex field equations, so it has a fundamental dynamical role in that theory [2]. For both theories, the continuity equation can be written using tensor notation as ($\mu = 0, 1, 2, 3$)

$$\partial_\mu J^\mu = \frac{\partial J^0}{\partial c_m t} + \frac{\partial J^x}{\partial x} + \frac{\partial J^y}{\partial y} + \frac{\partial J^z}{\partial z} = 0, \quad (11)$$

$$(J^\mu) = (\rho c_m, \rho u^x, \rho u^y, \rho u^z).$$

The first term in this expression for current $J^0 = \rho c_m$ allows for compressibility. We notice this rendition of the continuity equation treats spatial and temporal coordinates on an equal footing. This facilitates the integration of the space and time coordinates into a single spacetime geometry making it possible to develop a causal theory. In this form of the continuity equation, one can consider c_m to be present to rationalize the coordinates to length units. This has the result of making all components of the current have the same units of mass density \times speed.

From equation (11), when the spatial components of the current vanish, there remains a non-vanishing temporal component ρc_m . This is the speed in the time-direction for the fluid at rest: $J^k = 0$ and $J^0 = \rho c_m$. This quantity is also called the rest momentum density. Equation (11), written using a cartesian coordinate system, takes the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u^x}{\partial x} + \frac{\partial \rho u^y}{\partial y} + \frac{\partial \rho u^z}{\partial z} = 0 \quad (12)$$

on cancelling the constant factor c_m in the numerator and denominator of first term. This is the form conventionally used in fluid dynamics.

We can also consider the continuity equation in terms of the effects of a displacement of fluid at one point on the flow at another point. If current is to be conserved, its variation in one region is a source (a cause) for a reaction flow (an effect) in other regions of the flow. Both the energy and the momentum of the fluid flow are changed in both the source and in the other regions. *This is a global effect due to a local cause.* These long range effects, of course, do not occur at distant regions instantaneously. Thus the continuity equation constrains the dynamics of the flow in a pervasive, global manner. It is exactly this restriction on the flow by the continuity equation that requires the transverse modes [2].

2.2.4. Electromagnetic Lorentz force law and Poynting theorem. The Lorentz force law of electrodynamics gives the force f density acting on an electrical charge density ρ_D moving at a velocity u in the presence of an electric E and magnetic induction field B . It is stated in cartesian coordinate component form as (In gaussian, c.g.s., units, [30, see p.781 modifying section 6.7.):

Table 1. The isomorphism derived between the field equations of electromagnetic theory (EMT) and the field equations of the geometrodynamical theory of fluids (GTF).

EMT	GTF
$\nabla \cdot \mathbf{B} = 0$	(1) $\nabla \cdot \kappa \omega = 0$
$\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0$	(2) $\frac{1}{c_m} \frac{\partial \omega}{\partial t} + \nabla \times \frac{\lambda}{\kappa} \zeta = 0$
$\nabla \cdot \mathbf{D} = 4\pi \rho_D$	(3) $\nabla \cdot \bar{\zeta} = \frac{4\pi}{\bar{\eta}\lambda} c_m \rho$
$-\frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} + \nabla \times \mathbf{H} = \frac{4\pi}{c} \mathbf{J}_D$	(4) $-\frac{1}{c_m} \frac{\partial \bar{\zeta}}{\partial t} + \nabla \times \frac{\bar{\kappa}}{\lambda} \bar{\omega} = \frac{4\pi}{\bar{\eta}\lambda} \mathbf{J}$

$$f^i = \rho_D \left(E^i + \frac{1}{c} (u \times B)^i \right). \quad (13)$$

In equation (13), c is the speed of light which is the upper limit or maximum speed (phase velocity) of propagation of transverse electromagnetic waves in the medium. The charge density ρ_D in electrodynamics is a conserved quantity satisfying a continuity equation. The electrical current vector $(J^\mu) = (\rho_D c, \rho_D u^1, \rho_D u^2, \rho_D u^3)$ is a 4-vector also covariant under Lorentz transformations. The covariance arises essentially from the fact that ρ_D counts the number of charges in a unit volume.

For reference in the following discussion, we state Poynting's theorem which gives the differential form of the time-rate of change of energy and momentum due to an electromagnetic field interacting with electrical currents :

$$\begin{aligned} \frac{\partial \mathcal{E}_{EMF}}{\partial t} + \nabla \cdot \mathbf{S} &= -\mathcal{E}_{Joule}, \quad \mathbf{S} \equiv \frac{c}{4\pi} (\mathbf{E} \times \mathbf{H}), \\ \mathcal{E}_{EMF} &\equiv \frac{1}{8\pi} (\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}), \quad \mathcal{E}_{Joule} \equiv \mathbf{J} \cdot \mathbf{E}. \end{aligned} \quad (14)$$

Poynting's theorem, the first of equation (14), states a differential form of the conservation of energy. The quantity \mathcal{E}_{EMF} is the density of energy stored in the electromagnetic field. Poynting's vector \mathbf{S} defines the power flux: the energy crossing a unit area normal to the direction of the field propagation per unit time. In the definition of the Poynting vector \mathbf{S} , the vector \mathbf{E} is the electric field and \mathbf{H} is the magnetic field. In the definition of the energy \mathcal{E}_{EMF} , the vector \mathbf{D} is the electric displacement field and \mathbf{B} is the magnetic induction field. The current is denoted \mathbf{J} . The quantity $-\mathcal{E}_{Joule} > 0$ is the rate at which energy-density is produced by resistance heating.

2.2.5. Electrodynamics - fluid dynamics analogy. The fluid dynamical analog of the electromagnetic field, the GTF fluid dynamical vortex field or *vortex field* for short, is introduced in the beginning of this section. One expects such continuous physical systems to be described by similar field equations. That this is the case for GTF and electromagnetic theory (EMT) is shown in table 1 [2].

In table 1 the corresponding fields across the rows in the EMT and GTF columns are analogous. The standard meanings associated with electrodynamics are to be applied in the EMT column [30]. The field vectors with over-bars, $\bar{\omega}$ and $\bar{\zeta}$, termed *excitations*, are the response of the fluid to changes in ω or ζ , respectively. The quantities λ , κ , are linear *constitutive (material) parameters* which can be determined from laboratory experiments involving the generation and transmission of a fluid vortex field ($F_{\mu\nu}$) whose governing

equations are introduced below. For water, we estimate $\kappa \approx 1$, $\lambda \approx 0.01 \text{ (cm/s)}^{-1}$ [4]. From an experimental perspective, analysis of White's data for the flow of water in helical flow guides gives an estimated value of $c_m = 150 \text{ m s}^{-1}$ [22]. This value is many orders of magnitude smaller than the value for the maximum speed of transverse waves c used in electrodynamics, i.e., $c_m \ll c = 29.979 \times 10^9 \text{ cm s}^{-1}$. The choice of which combination of the material parameters $\{\bar{\eta}, \bar{\lambda}, \lambda, \kappa, \bar{\kappa}, c_m\}$, of course, should be determined by a combination of experimental methodology and theoretical considerations. For instance, for a linear constitutive theory, the number of parameters can be reduced by dimensional analysis as shown in the companion paper where we show the material constants $\bar{\kappa}\bar{\lambda}$, $\kappa\lambda$, and $\bar{\eta}\bar{\lambda}$ can be replaced by $\kappa = 1$, and $\bar{\eta}\bar{\lambda} = c_m$.

In interpreting the GTF column of table 1, note we have chosen the nomenclature 'vorticity' and 'swirl' and the notation ω and ζ to distinguish these new fields from the older Navier-Stokes fields of vorticity (pseudo-vector), denoted $\Omega = \nabla \times u$, and the Navier-Stokes swirl or Lamb vector $\Lambda = \Omega \times u$. Physically, the GTF vorticity ω is similar to the NST vorticity $\Omega = \nabla \times u$. The major difference in meaning between the two is the NST quantity Ω is simply a mathematical transformation (taking the curl) of the velocity vector u . In comparison, the GTF vorticity (pseudo-) vector ω introduces a new field and new physics to the description of fluid flow whereas the NST quantity Ω does not. The GTF swirl ζ , also a new (vector) field, is different from the Lamb vector $\Lambda = \Omega \times u$ in that the Lamb vector is similarly a derived quantity. From the analogy, the swirl field ζ can be expected to be important for time-varying flows or compressible fluids. The swirl field ζ has the effect of causing fluid flow to follow along its generally curved lines of force, no matter what the fluid inertia might be, hence its name 'swirl.' In distinction to the NST variables, both ω and ζ are elements of the second-order vortex field tensor $F_{\mu\nu}$; they describe the enhanced vortical and swirling motion of fluid flow often observed in the experiment.

The similarities GTF and EMT are clear: both the fluid theory and the electrodynamical one describe current flows defined over a 4D spacetime. In both theories, a continuity equation $\partial_\mu J^\mu = 0$ is satisfied. For a strict analogy to exist, the spacetime must be of the same mathematical structure. Since electrodynamics is formulated over a 4D Minkowski spacetime, for a fruitful correspondence, the fluid dynamical theory must be formulated for an analogous spacetime. The electrodynamical theory describes the evolution of a current J which is a 4-vector defined over a hyperbolic spacetime with a maximum speed of transverse waves being the speed of light c . That is, J is a 4-vector with one time and three spatial components. The fluid analog for incompressible fluids is $J = (\rho_0 c_m, J^1, J^2, J^3)$ in a cartesian coordinate system. More generally, for compressible fluids, the current can be defined in terms of the covariant derivative of the vortex field $J^\mu = (c/4\pi)F^{\mu\nu}{}_{;\nu}$ as shown in the companion paper [1]. This definition renders the fluid dynamical current a c_m -Lorentz covariant 4-vector. The proof of this statement involves the introduction of the vortex field theory, in particular, what we call the vortex field lemma (VFL, [2]). For speeds $\|u\| \ll c_m$ this definition limits to $(J^\mu) = (\rho c_m, \rho u^1, \rho u^2, \rho u^3)$. Since c_m has units of velocity, the units of ρc_m and ρu^k are the same, so all components of J have the same units as required. Here we introduce coordinates $(x^\mu) = (x^0, x^1, x^2, x^3)$ and describe the geometry using a metric tensor $(g^{\mu\nu}) = \text{diag}(1, -1, -1, -1)$ in units where $c_m = 1^3$. For cartesian coordinates; $(x^\mu) = (c_m t, x, y, z)$. For simplicity, as in electrodynamics, theoretical expressions involving c_m are often written as if $c_m = 1$.

To be causal theories, both electrodynamical and fluid dynamical field theories should be defined over a 4D Minkowski spacetime. Limitations then exist on the kinds of fields that can

³ We follow the metric convention for matter fields of [21, 30], and [19]: $(g^{\mu\nu}) = \text{diag}(1, -1, -1, -1)$.

play a part. Such field theories are limited to containing scalar, vectors, second order tensor, pseudo-vector, and pseudo-scalar fields [19]. The scalar of electrodynamics is the charge density ρ_D . It is a scalar, and not a pseudo-scalar since it does not change sign if all spatial coordinates are sign-reversed. We have just introduced the vector of the theory, the current J . It is a product of a scalar and an ordinary vector, the velocity. Under coordinate reversals, the current vector J changes sign. The distinction between vectors and pseudo-vectors also arises from their transformations under reversals of spatial coordinate directions. In electrodynamics the field $B = \nabla \times A$ is not a vector, rather it is termed a pseudo-vector field since changing all three spatial coordinates reverses the sign of both ∇ and the vector potential A leaving the sign of B unchanged. In electrodynamics, there is introduced the electromagnetic field tensor $F_{\mu\nu}$ which is a second order tensor consisting of a vector part (the electric field E) and a pseudo-vector part B , the magnetic field). The current vector J is responsible for the generation of $F_{\mu\nu}$ [2].

To develop a fluid dynamical theory with this field structure, the fluid analog of the electromagnetic field, the GTF vorticity ω - and swirl ζ - fields, are expressed in terms of a pseudo-vector and a vector field. The vorticity ω and swirl ζ then become components of a fluid *vortex field tensor* $F^{\mu\nu}$. This tensor unifies vorticity ω and swirl ζ into a single second order tensor quantity [2]. We have chosen the symbol for the vortex field tensor $F_{\mu\nu}$ to be the same as that of the electromagnetic field tensor to emphasize the isomorphism of the fluid vortex field equations to the electromagnetic field equations. (See equations (15)–(16) and table 1 above.)

The analogy can be cast into a simple form as a linear one-to-one map between the variables of one theory to those of another. For a cartesian coordinate system, the analogy consists of identifying the electromagnetic field components (E, B) of $F_{\mu\nu}$ with the fluid vortex field components (ζ, ω) of the fluid $F^{\mu\nu}$ as follows

$$\begin{aligned} E_x &\simeq \lambda\zeta_x, & E_y &\simeq \lambda\zeta_y, & E_z &\simeq \lambda\zeta_z, \\ B_x &\simeq \kappa\omega_x, & B_y &\simeq \kappa\omega_y, & B_z &\simeq \kappa\omega_z. \end{aligned} \quad (15)$$

This arrangement states a linear one-to-one mapping between the two sets of fields. The analogy between the GTF and EMT can be displayed succinctly by the map between the vortex field components:

$$\left(F_{\mu\nu} \right) = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \simeq \begin{pmatrix} 0 & \lambda\zeta_x & \lambda\zeta_y & \lambda\zeta_z \\ -\lambda\zeta_x & 0 & -\kappa\omega_z & \kappa\omega_y \\ -\lambda\zeta_y & \kappa\omega_z & 0 & -\kappa\omega_x \\ -\lambda\zeta_z & -\kappa\omega_y & \kappa\omega_x & 0 \end{pmatrix} \quad (16)$$

Here the $(\mu, \nu)^{th}$ element of $\left(F_{\mu\nu} \right)$ on the left is analogous to the corresponding element on the right-hand side for the cartesian coordinate system. As explained above, the GTF fields ω and ζ on the right are new to fluid dynamical theory. Physically, they account for additional modes of storage and transmission of energy and momentum in the fluid. The fact $\left(F_{\mu\nu} \right)$ is a second-order tensor implies the (E, B) -field pair and analogously the (ζ, ω) -field pair are each interrelated. The (ζ, ω) -fields are *generated* by the flow of the fluid in exactly the same (mathematical) way the (E, B) -fields are generated by the ‘flow’ of electrical current [2].

3. Fluid Dynamical Lorentz Force Law and Poynting Theorem

Having explained the analogy presented in equation (16) or table 1, we are ready to apply it to the electro-dynamical Lorentz force law, equation (13). The fluid dynamical analog of the Lorentz force (density) law is then found:

$$f^i = \lambda J_0 \zeta^i + \kappa (J \times \omega)^i = J_0 \left(\lambda \zeta^i + \frac{\kappa}{c_m} (u \times \omega)^i \right). \quad (17)$$

Here $J_0 = \rho c_m$ and in the second equation we use $J^i = \rho u^i$, a relation valid for $\|u\| \ll c_m$. In electromagnetic theory, the Lorentz force can be derived from the stress-energy flux of the electromagnetic field. By comparing the expression for the fluid Lorentz force above to the electromagnetic case given in equation (13) suggests the constant κ can be set to unity. The significance of the Lorentz force generated by the transverse wave modes is that it contributes to the balance of stress-energy flux of a continuum [2].

The swirl field ζ produces a force-density $f^i = \lambda J_0 \zeta^i$ impelling fluid particles to move along (or rotate about and translate along if $\omega \neq 0$) the ζ field lines. We find the units $[\lambda] = T/L$, so in terms of units $[\lambda c_m] = 1$, i.e., λc_m is unitless. Thus, the fluid dynamical ρ corresponds to the electromagnetic charge density. In analogy to the electro-dynamical case where a steady, uniform charge density creates a constant background electric field, the average fluid density is responsible for a steady, longitudinal ζ -mode of the vortex field.

Nonlinear effects originate from several sources. For a *compressible* fluid, the term $J_0 = c_m \rho$ is spatially and temporally varying, so the *product* $\lambda J_0 \zeta^i$ of the constitutive parameter λ with J_0 , and the swirl components ζ^i makes a *nonlinear* contribution to the force acting on fluid elements. This variation is a source for the swirl field ζ^i in the first term $\lambda J_0 \zeta^i$ on the right-hand side. For an incompressible flow J_0 is a constant, so this nonlinear effect is absent. The second term in equation (17) $\kappa (J \times \omega)^i$ describes a nonlinear interaction of the current with the vorticity field ω . This term couples the fluid current J nonlinearly, because of the cross *product* $u \times \omega$ of the velocity u with the vortex field vorticity ω . This determines a functional dependence of the vorticity on the velocity, denoted by $\omega[u^i]$. Consequently, the term $(J \times \omega)^i$ in the field equations for the velocity produces a non-linear velocity-velocity interaction. Such nonlinearities allow one mode to interact with another mode, i.e., allow mode coupling. (Another source of nonlinearity in fluid flow is the directional derivative term, $u \cdot \nabla u^i$, in the balance of stress-energy flux equations.) The second term in equation (17) causes fluid particles to orbit the vorticity ω field lines.

Applying the analogy presented in equation (15) to equation(17), we obtain the fluid dynamical Poynting theorem:

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \vec{S} = -\lambda c_m J \cdot \zeta, \quad (18)$$

$$S^i = \left(\frac{c_m \bar{\eta}}{4\pi} \right) (\lambda \zeta \times \kappa \omega)^i, \quad \mathcal{E} = \frac{\bar{\eta}}{8\pi} (\lambda^2 \zeta^2 + \kappa^2 \omega^2).$$

Here the S^i are components of the fluid dynamical Poynting vector and \mathcal{E} is the energy-density of the fluid vortex field. (Analysis of the companion paper [1] shows the Poynting vector is actually not a 4-vector, since \vec{S} is composed of the time components of a second-order tensor.) According to the second part of equation (18), the Poynting vector is orthogonal (transverse) to the vorticity and swirl. The expression for \mathcal{E} given here is the for a transverse wave field. The first of equation (18) describes the balance of energy-momentum flux. To see this, let us divide the equation by c_m :

$$\frac{\partial \mathcal{E}}{\partial c_m t} + \frac{1}{c_m} \nabla \cdot \vec{S} = -\lambda J \cdot \zeta, \quad (19)$$

Equation (18) gives an energy-density representation. Since \vec{S} is an energy density, dividing it by c_m gives a momentum density so equation (19) involves a energy-momentum 4-vector. Then consider this energy-momentum 4-vector $S_c \equiv (\mathcal{E}, \vec{S}/c_m) = (S^0, S^1, S^2, S^3)$. Using the tensor notation for a 4D spacetime introduced above, we can express the Poynting theorem as the 4-divergence $\sum_{\mu=0}^3 \partial_{\mu} S_c^{\mu}$ of the vortex field energy-momentum vector. Then, for a unit volume $\partial_{\mu} S_c^{\mu}$ is equal to the work per unit time (power) done by the swirl field on the current $-\lambda J \cdot \zeta$. Thus, as in electrodynamics, the Poynting theorem describes the transfer of energy-momentum. The term $-\lambda J \cdot \zeta$ is seen to be a maximum when the swirl field ζ is aligned parallel to the current J . This maximum occurs when the current slides along the swirl field. The analogous term for the electromagnetic field is the term to the right-hand side of equation (19), $-J \cdot E$. This is the density of electrical power dissipated by the Lorentz force acting on the (electrical) current. It gives the ‘resistive’ energy losses. (So-called I^2R losses leading to Joule heating.) In the fluid case, this energy loss is termed a loss due to frictional forces. Thus, by analogy, the term $-\lambda J \cdot \zeta$ can be interpreted as providing energy dissipation identical to that ascribed to the newtonian viscous stresses $(\eta/\rho_0)\nabla^2 u$ due to fluid elements sliding past one another. For example, suppose ζ is a longitudinal field down a flow guide causing a steady flow. Then $\lambda J \cdot \zeta = \lambda \rho_0 u \cdot \zeta$ gives the energy dissipation reflecting the velocity profile u to the same extent as the viscous dissipation defined by the Navier-Stokes equations.

By writing the fluid dynamical Poynting theorem as

$$\frac{\partial \mathcal{E}}{\partial t} = -\lambda c_m J \cdot \zeta - \nabla \cdot \vec{S}, \quad (20)$$

it is seen the energy density dissipation rate of the vortex field consists of an energy density loss rate $-\lambda c_m J \cdot \zeta$ creating heat and an energy flux loss rate $-\nabla \cdot \vec{S}$ or momentum flux $-\nabla \cdot \vec{S}/c_m$. Using Stokes’ theorem and the non-slip boundary condition, the integral of the momentum transfer over a section of a flow guide reduces to a quantity proportional to the difference between the outlet and inlet values of \vec{S} projected onto the flow guide cross section:

$$\int_{M^3} d^3x \nabla \cdot \vec{S} = \int_{\partial M^3} (\vec{S}_{out} - \vec{S}_{in}) \cdot d\vec{A}. \quad (21)$$

The momentum transfer $(\vec{S}_{out} - \vec{S}_{in})/c_m$ is experimentally measurable in terms of the force that must be applied to the flow guide to keep it stationary. The excitation of a new vortex field mode would change this force. Again this is an experimentally measurable effect. The change in energy is given by

$$-\frac{d}{dt} \int_{M^3} \mathcal{E} d^3x = \int_{M^3} \lambda c_m J \cdot \zeta d^3x + \int_{\partial M^3} (\vec{S}_{out} - \vec{S}_{in}) \cdot d\vec{A} \quad (22)$$

This shows the total change in energy also includes a dissipative part $J \cdot \zeta$.

In summary, the viscous stress used in the NSEs can be said to lead to a kinetic energy dissipation *channel*. The results of equations (20)–(22) then introduce *new channels* of energy-momentum propagation and dissipation due to transverse fluid flow modes replacing the newtonian viscous stress channel. Thus “viscosity” and energy-dissipation has not been eliminated. They have reappeared in another form described by the Poynting theorem

4. Consequences

In this section the combined GTF equations are given for the low speed limit $\|u\|/c_m \ll 1$. We have argued that the Poynting theorem accounts for the stress-energy propagation and dissipation in a viscous fluid. For simplicity, we consider the constant density, small flow speed limit (relative to the maximum transverse mode speed c_m , i.e., $\|u\|/c_m \ll 1$) of the stress-flux balance. This limit yields the following set of equations for the velocity field components u^i for an incompressible fluid [4, 21]

$$u \cdot \nabla u^i + \frac{1}{\rho} \nabla^i p = \frac{\eta}{\rho} \nabla^2 u^i, \quad (23)$$

$$\frac{\partial u^i}{\partial t} + u \cdot \nabla u^i + \frac{1}{\rho} \nabla^i p = \lambda c_m \zeta^i + \kappa (u \times \omega)^i, \quad (24)$$

$$\begin{aligned} \nabla \cdot \omega &= 0, & \frac{1}{c_m} \frac{\partial \omega}{\partial t} + \nabla \times \frac{\lambda}{\kappa} \zeta &= 0, \\ \nabla \cdot \bar{\zeta} &= 4\pi\rho, & -\frac{1}{c_m} \frac{\partial \bar{\zeta}}{\partial t} + \nabla \times \frac{\bar{\kappa}}{\lambda} \bar{\omega} &= \frac{4\pi}{c_m} J. \end{aligned} \quad (25)$$

Equation (23) gives the equation of stress-balance for time-independent flow from the Navier-Stokes equations. For time-independent flows, there is no acausality concern with action-at-a-distance. The solutions of the time-independent NSEs can be obtained for any time without reference to the flow at any other time—solutions at any time are identical. For such time-independent flows equation (23) can be non-dimensionalized so the right-hand side takes the well-known form $\text{Re}^{-1} \nabla_*^2 u_*^i$ where $\text{Re} = UL/(\eta/\rho)$ where U is a characteristic velocity of the flow and L a characteristic length scale; $\mu = \eta/\rho$ is the kinematic viscosity. As Re increases without bound, the newtonian viscous stresses become increasingly inconsequential compared to the inertial terms on the left-hand side. In this régime, the inertial forces have no counterbalance. Equation (23) is clearly limited to low Re , time-independent flow. In equation (25), the excitations ($\bar{\omega}$, $\bar{\zeta}$) are related to the vortex fields (ω , ζ) via constitutive relations: $\bar{\omega} = \bar{\lambda}^{-1} \omega$, $\bar{\zeta} = (\lambda \bar{\lambda}^{-1}) \zeta$, with $c_m^2 = \kappa \bar{\lambda} / \bar{\kappa} \lambda$.

For the GTF, on the other hand, equation (24) shows the Lorentz force $f^i = F^{iv} J_v$ counterbalances the inertial force. The Lorentz force term leads to enhanced mixing and stress-energy transport (via convection, vortex field excitation, and propagation) in the fluid compared to the NST. Stress-energy dissipation is also enhanced according to the Poynting theorem because of the ‘Joule’ heating term $\lambda J \cdot \zeta$. The left-hand side of equation (24) is obtained from the inertial stress-energy in the low speed limit $\|u\|/c_m \ll 1$ as described in [21]. An important point is the pressure term $\rho^{-1} \nabla^i p$ is considered as part of the inertial forces of the fluid in GTF. This arises from the computation of the total stress-energy of the fluid. The term $-(\eta/\rho) \nabla^2 u^i$ due to newtonian viscous stresses in equation (23) is omitted in equation (24) as required to avoid acausality. The covariant 4-current J exciting the vortex fields is obtained from the vortex field as $J^\mu \equiv (4\pi)^{-1} \bar{\eta} F^{\mu\nu}{}_{;\nu}$. This current has to be determined simultaneously from the set of equations (24), (25). For higher speed flows the full theory described in the companion paper is required.

5. Summary and conclusions

This paper provides a physically motivated introduction to the fluid dynamical Lorentz force and Poynting theorem and a description of how they fit into the larger problem of devising a covariant, causal field theory of fluid flow. The exposition is based on the isomorphism of the fluid dynamical and eletrodynamical vortex field equations proven in the companion paper [1]. This allows the introduction of a fluid dynamical/electrodynamical analogy. First we give an analysis of the Navier-Stokes theory (NST). We show the NST is an acausal theory supporting action-at-a-distance with infinite speeds of propagation of velocity disturbances. This is characteristic of physical theories formulated for galilean spacetimes. The newtonian viscous stress term is the source of the physical acausality; mathematically, the newtonian viscous stresses introduces the galilean spacetime rendering the NSEs as parabolic partial differential equations. The parabolicity leads to the infinite speeds of velocity disturbance propagation typical of the (parabolic) diffusion equation. Experimental evidence is cited showing the maximum speed of transverse mode propagation c_m in fluid flow is, in fact, substantially less than the speed of light.

The new geometrodynamical theory of fluids (GTF), through the aegis of the Poynting theorem, demonstrates the existence of new channels of energy-dissipation and transport that can account, for instance, for the large increase of energy-dissipation measured in helical flows on excitation of transverse flow modes. The Lorentz force for fluid dynamics provides a physical description of the excitation of these transverse modes.

The GTF solves the acausality and speed problems of the NST by introducing a Minkowski spacetime with a finite maximum speed of propagation c_m . This replaces the galilean spacetime used for the NST where the speed of propagation is infinite. The GTF solves the problem of anomalous stress-energy dissipation by introducing new avenues of stress-energy propagation and dissipation that are described by the fluid dynamical Lorentz force and by the fluid dynamical Poynting theorem, respectively. The Lorentz force induces new modes of flow associated with the sliding of fluid lamella past one-another. This leads to energy-dissipation analogous to Joule heating in electrodynamics described by the Poynting theorem. The fluid dynamical Lorentz force depends on the fluid GTF vorticity ω and GTF swirl ζ fields in the same way as the electrodynamical one depends on the magnetic B and electric E fields. Together the ω and ζ fields constitute the independent components of the fluid vortex field tensor, the analog of the electromagnetic field tensor $F_{\mu\nu}$.

Within the GTF, a covariant 4-vector current based on the vortex field is defined. The resulting theory is c_m -Lorentz covariant, meaning the equations describing the fluid flow are invariant under spacetime transformations keeping c_m constant for fluids and observers in relative motion. The theory is not restricted to small flow speeds: as the flow speed increases, the inertial forces remain in balance with the fluid dynamical Lorentz forces. This leads to increased stress-energy dissipation; this is unlike the case of the NST where inertial forces become unbalanced as the newtonian viscous stresses become relatively small, as the flow speed increases.

For lower speeds satisfying $\|u\| \ll c_m$, we introduce the stress-energy flux balance equations to relate these results. These equations resemble the Navier-Stokes equations in that the forces due to the inertial stress-energy are present. However, the force due to newtonian viscous stress is replaced by the Lorentz force for time-dependent flows. This strikes a new stress-energy flux balance for describing dissipative fluid flow. Our results are rigorously derived in the companion paper [1].

Examples are also cited of fluid vortex field modes observed in experiment and compared to their electromagnetic analogs. Because helically curved flow guides limit the wavelength of

the vortex field modes and their (maximum phase) speed of propagation, such guides can be usefully employed to experimentally determine the constitutive parameters and wave speeds of the fluid vortex field. Measurement of the mode structure, mode amplitude, energy-dissipation, and comparison to theory can be used to obtain the constitutive parameters and their frequency and wave vector dependence. The present theory and future experiments are expected to improve the description of time-dependent and turbulent flows where stream tubes of curved geometry are prevalent.

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