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# Fluid dynamical Lorentz force law and Poynting theorem—derivation and implications

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**Abstract**
Fluid dynamical analogs of the electrodynamical Lorentz force law and Poynting theorem are *derived* and their implications analyzed. The companion paper by Scofield and Huq 2014 *Fluid. Dyn. Res.* **46** 055513 gives a heuristic introduction to the present results. The fluid dynamical analogs are consequences of a new causal, covariant, geometrodynamical theory of fluids (GTF). Compared to the Navier–Stokes theory, GTF shows the existence of new causal channels of stress-energy propagation and dissipation due to the action of transverse modes of flow. These channels describe energy-dissipation and transport along curved stream tubes common in turbulent flows.

## 1. Introduction

This paper presents a *rigorous derivation* of the fluid dynamical Lorentz force law, the fluid dynamical Poynting theorem, and an analysis of their implications for the theory of fluids. In so doing, this paper describes a solution to the problem of formulating a covariant, causal theory of time-dependent fluid flow [2]. This theory is called the geometrodynamical theory of fluid flow (GTF, [3]). From the GTF equations we derive the fluid dynamical Poynting theorem for the transfer of stress-energy. This shows the stress-energy transfer occurs via finite-speed transverse modes. These modes are excited by the fluid dynamical Lorentz force. The modes lead to new channels of stress-energy dissipation absent from the Navier-Stokes theory. The companion paper [1], provides a *heuristic introduction* to the Lorentz force law and Poynting theorem using an analogy to electromagnetic theory [4] for expository purposes. Previously, such electrodynamical analogies have been used to better understand time-dependent flow; in particular in [5–7] and more recently by Kambe [8]. Kambe's work is to be noted for its rigor and for its introduction of a gauge theoretic perspective into such analogies [9]. The present work differs from these since it is not based on analogy; as we show, it is based on the mathematical consequences of the GTF equations [3]. The theory can describe time-dependent, high speed flows for which inertial forces are balanced by Lorentz forces along with enhanced stress-energy dissipation without the introduction of eddy viscosity.

The present paper is organized as follows. The background summarizes the physical basis of the theory developed in the companion paper and compares it to the isomorphic theory of electromagnetism. We then *derive* the fluid dynamical Lorentz force and Poynting theorems from the fluid dynamical *vortex field tensor*, the fluid dynamical analog of the electromagnetic field tensor. The consequences of the Lorentz force and Poynting theorems are then developed by formulating the whole set of geometrodynamical theory of fluids (GTF) equations. This is followed by an analysis of the inclusion of the newtonian viscous stresses to assess how these stresses compare to the new channels of stress-energy provided by the vortex field. A summary and conclusion section follows. The first appendix gives the definitions of the tensors used in the formal derivations. The second appendix shows the formulation presented here can be expressed in a way that the GTF vorticity $\omega$ and swirl $\zeta$ fields have the same units as the vorticity $\Omega = \nabla \times u$ and Lamb $\Lambda = \Omega \times u$ vectors have in the Navier-Stokes theory (NST).

## 2. Background

In the companion paper we show the need to revisit the foundations of fluid mechanics arises from complications due to the Navier-Stokes equations being parabolic partial differential equations (PDEs) rather than being $c_m$-Lorentz covariant hyperbolic PDEs. Their diffusion equation formulation, with an attendant infinite speed of velocity propagation, implies action-at-a-distance, a formulation that is termed acausal. On the other hand, a finite speed of propagation of signals allows causes and effects at any field point to be sequentially ordered by arrival time. An infinite speed of propagations leads causes and effects to be simultaneous. In the companion paper we discuss the fact the Navier-Stokes equations (NSEs, [2]) are *acausal*. The NSEs embody action-at-a-distance where all causes arrive from infinitely distant places simultaneously but they are *not* non-causal where effects can precede causes. This action-at-a-distance is a characteristic of newtonian physics where the speed of all signal propagation is infinite. We also point out for time independent flows, because there is no change, that there is no propagation at any speed. In this case, the Navier-Stokes equations reduce to elliptic PDEs having laminar flow solutions.

The problems of the non-relativistic NSEs extend to the relativistic formulation of fluid dynamics given by Landau and Lifshitz [2]. Their fluid theory is an example of a covariant, acausal theory holding the speed of light $c$ constant. It is presently the standard theory of relativistic fluids. The acausality of this theory [10] has been discussed extensively [11–17, 19]. This analysis concludes, given there was no alternative theory at the time, that the Landau and Lifshitz formulation was adequate as long as the fluid could relax sufficiently fast enough to mask the acausality. This work also shows a finite speed of wave propagation is a necessary (but not sufficient) ingredient of causal theories.

The geometrodynamical theory of fluids (GTF) given in [3] shows it is possible to avoid these shortcomings by using a theory based on the geometrodynamics of current conserving spacetimes with finite speeds of transverse wave propagation. The GTF introduces Lorentz forces and Poynting theorems for both fluid dynamics and electrodynamics. In the companion paper we describe the resulting, causal theory of fluid mechanics (GTF) by relating it to the analogous theory of electromagnetism (EMT). Each of these theories respect a maximum speed of wave propagation that depends on the material medium. Both theories are causal theories and can be expressed in a covariant form where the basic equations of the theory are form invariant under transformations of coordinates. Both are also Lorentz-covariant, meaning the form of the equations are invariant with respect to transformations of the spacetime coordinates even including coordinate systems moving at constant relative speeds to the flow. Both theories involve covariant 4-currents. We describe the calculation of the fluid 4-current in this paper.

Integral to the GTF theory is the existence of a maximum speed of transverse waves, denoted $c_m$ and two other phenomenological constitutive parameters that we discuss in the next section. For a fluid continuum a finite speed of propagation is also required to escape the conundrum of newtonian physics action-at-a-distance and infinite speeds of transverse mode propagation characteristic of the NST. Avoiding this problem is required if one is to consistently combine transverse mode propagation (fluid dynamical Lorentz forces) and covariant stress-energy flux balance. Experimental measurements show the maximum speeds of fluid dynamical waves $c_m$ is of an order $10^{-5}$ times smaller than the maximum speed of propagation of light waves in empty space, the speed of light $c = 2.9979 \times 10^8$ m s$^{-1}$, [20–22]. Because of this limitation, the geometry of the spacetime required for fluids acting under a fluid vortex field must employ the smaller by $10^{-5} \times$ speed of propagation. The questions raised by the analysis of [11–19], then implies relaxation times are no longer relatively short, so an alternative theory to the relativistic NST of [2] needs to be formulated (e.g., GTF).

In summary, causality is physically related to the existence of a maximum speed of signal propagation and mathematically to the use of hyperbolic, second-order wave equations with a single time-like variable. In a sense, it is remarkable that finite speed of signal propagation, causality, and spacetime geometries are so intimately related. In the following parts of this paper, by introducing a finite speed of propagation $c_m$, a covariant, causal theory of fluids is formulated that is expressible in terms of hyperbolic wave equations and covariant stress-energy flux balance, —the GTF theory. This enables the derivation of fluid dynamical Lorentz forces and energy transport described by Poynting's theorem.

### *2.1. Vortex Field Equations*

Causal theories such as electrodynamics address the foundational problems of the propagation of stress and energy in a continuum. They require a finite speed of propagation. This allows the concept of 'transport' of causes to effects to be meaningfully defined. In this we can include in 'transport' a combination of convection and propagation. The vortex field equations introduced immediately below are causal equations formingg part of the GTF. As explained in the companion paper, these equations allow one to describe the propagation, not acausal diffusion, of the velocity, vorticity, and swirl fields contributing to the stress-energy flux balance in a fluid. In the remainder of this section, we will give a synopsis of the vortex field equations and the terms used to describe the balance of stress-energy flux in a $c_m$-spacetime. The basic result is the following:

**Theorem 1.** *Vortex Field Equations (VFEs, Scofield-Huq, [3]). Consider a simply connected 4D $c_m$-ST manifold and a homogeneous, isotropic fluid having a linear constitutive relation between its vortex field $F(\zeta, \omega)$ and its excitations $H(\bar{\zeta}, \bar{\omega})$, $H_{\kappa\lambda} = C^{\mu\nu}_{\kappa\lambda} F_{\mu\nu}$. Then the conservation of the 4-current $J$ for homogeneous isotropic media implies the*

**Table 1.** The isomorphism derived between the vortex field equations of electromagnetic theory (EMT) and geometrodynamical theory of fluids (GTF).

| EMT | | GTF |
|---|---|---|
| $\nabla \cdot B = 0$ | (1) | $\nabla \cdot \kappa\omega = 0$ |
| $\frac{1}{c}\frac{\partial B}{\partial t} + \nabla \times E = 0$ | (2) | $\frac{1}{c_m}\frac{\partial \omega}{\partial t} + \nabla \times \frac{\lambda}{\kappa}\zeta = 0$ |
| $\nabla \cdot D = 4\pi\rho_D$ | (3) | $\nabla \cdot \bar{\zeta} = \frac{4\pi}{\bar{\eta}\bar{\lambda}} c_m \rho$ |
| $-\frac{1}{c}\frac{\partial D}{\partial t} + \nabla \times H = \frac{4\pi}{c} J_D$ | (4) | $-\frac{1}{c_m}\frac{\partial \bar{\zeta}}{\partial t} + \nabla \times \frac{\bar{\kappa}}{\bar{\lambda}}\bar{\varpi} = \frac{4\pi}{\bar{\eta}\bar{\lambda}} J$ |

*geometrodynamical theory of fluid (GTF) vortex field equations and an isomorphism with electromagnetic theory (EMT) as displayed in table 1.*

The isomorphism given in table 1 illustrates the profound consequences of the conservation of currents [3]; the physical currents in EMT and GTF are quite different, yet they obey isomorphic vortex field equations. Table 1 illustrates the physical theory analogy. However, the correspondence is deeper. Both EMT and GTF are derived from the vortex field lemma, a consequence of the conservation of currents (electrical and fluid, respectively). Here $\rho_D$ is the electrical charge density and $J_D$ is the electrical current, the electric field vector is denoted by $E$ and the magnetic field vector by $B$. Their excitation fields are the vectors $D$ and $H$. The parameters $C^{\mu\nu}_{\kappa\lambda}$ in the theorem statement are linear constitutive (material) parameters of the vortex fields—different from the viscosity parameters of the NST. They relate the vortex field strength $F(\zeta, \omega)$ to its 'excitations' $H(\bar{\zeta}, \bar{\varpi})$ in the same way the field strengths $F(E, B)$ and excitations $H(D, H)$ are related in electrodynamics [23]. (See ref. [4, section 6.8–9] for the analogous case where lossy electromagnetic media is also considered.) The excitations $H(\bar{\zeta}, \bar{\varpi})$ are related to the fields $F(\zeta, \omega)$ in that the excitations are the response of the system to field variations. The excitations of the *vorticity* $\omega$ and *swirl* $\zeta$ fields are denoted with an over-bar $(\bar{\zeta}, \bar{\varpi})$. The $(\bar{\zeta}, \bar{\varpi})$ are not 'fluctuations' about a mean. Thus these quantities are not directly related to the Reynolds decomposition of mean quantities compared to their perturbations often used in the analysis of turbulent flow. The variation of the density $\rho$ allows one to define the quantity $\delta\rho = \rho - \rho_{avg}$ as the fluid density fluctuation about the average value, $\rho_{\text{avg}}$; it is also not a Reynolds decomposition. The spatial current components are given by $J^i = \rho u^i$. In the absence of longitudinal fields due to $\rho_{avg}$ so $\nabla \cdot \bar{\zeta} = \delta\rho = 0$, there is still an analog of the electrical displacement field ($D \simeq \bar{\zeta}$) whose time variation $\partial\bar{\zeta}/\partial t$ produces a curl of the analog of the vorticity excitation $-\frac{\bar{\lambda}}{c_m}\frac{\partial \bar{\zeta}}{\partial t} + \bar{\kappa}\nabla \times \bar{\varpi} = \frac{4\pi}{\bar{\eta}} J$.

Transverse waves are predicted by the equations of table 1 given appropriate geometric constraints as in electrodynamics [1–4]. For instance, transverse vorticity (TV) waves are predicted by solving these equations for flow along flow guides of constant cross-section and vanishing velocity along the walls of the guide. Transverse swirl (TS) waves have swirl components transverse to the direction of propagation. Transverse vorticity-swirl (TVS) waves are predicted for propagation in unbounded media. These TV, TS and TVS modes correspond to the TM, TE and TEM modes of electrodynamics, respectively.

The GTF formulation given in table 1 is general so that we can flexibly determine the physical units and constitutive relations depending on experimental methodology and theoretical requirements. Using the flexibility of four constitutive parameters, in appendix B, we show the equations can be simplified using dimensional analysis so only a total of three material parameters are needed for the whole set of equations by choosing the units of GTF vorticity $\omega$ and swirl $\zeta$ to have the same units as the corresponding quantities appearing in the Navier-Stokes theory $\Omega = \nabla \times u$ and $\Lambda = \Omega \times u$. This allows direct comparison of results of the GTF with the NST field variable pairs $(\omega, \zeta)$ and $(\Omega, \Lambda)$. We show, one can consistently set $\kappa = 1$ and $\bar{\lambda}\bar{\eta} = c_m$. This results in $c_m^2 = 1\big/(\bar{\kappa}\lambda\bar{\lambda}^{-1})$ and the elimination of the quantity $\bar{\eta}$ scaling the coupling of the fluid current to the fluid vortex field. The minimal set equations are given in equation (B.10) of appendix B.

From our understanding of EMT (electrodynamical vortex field equations), we can ascertain some of the physical meaning of the GTF equations in table 1 (fluid dynamical vortex field equations). First $(\zeta, \omega)$ and $(\bar{\zeta}, \bar{\omega})$ represent the fields and their linear excitations [23]. Second, as shown immediately below, the fields satisfy second-order vector wave equations sourced by the fluid current and density fluctuations. The solution of these equations leads to the introduction of transverse and longitudinal modes of flow exactly analogous to those of electrodynamics. The analogy *derived* from the vortex field lemma (VFL, see appendix A.) can be neatly arranged in terms of an equivalence of the electromagnetic $\left(F_{\mu\nu}\right)$ and the fluid $\left(F_{\mu\nu}\right)$ matrices (metric $(g^{\mu\nu}) = \mathrm{diag}(1, -1, -1, -1)$ using 'theoretical' units where $c = 1$ and $c_m = 1$):

$$\left(F_{\mu\nu}\right) = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \approxeq \begin{pmatrix} 0 & \lambda\zeta_x & \lambda\zeta_y & \lambda\zeta_z \\ -\lambda\zeta_x & 0 & -\kappa\omega_z & \kappa\omega_y \\ -\lambda\zeta_y & \kappa\omega_z & 0 & -\kappa\omega_x \\ -\lambda\zeta_z & -\kappa\omega_y & \kappa\omega_x & 0 \end{pmatrix} \tag{1}$$

The VFEs of table 1, show the 4-vector current $(J^\mu) = (c_m\rho, J^x, J^y, J^z)$ is the source driving the excitation of the vorticity and swirl components of the vortex field $F^{\mu\nu}$. Mixed Dirichlet–Neumann boundary conditions can be used: specifying values for $\zeta$ and normal derivatives of $\omega$ on boundaries, in solving the field equations. Overall, the units are chosen so that $\omega$ and $\zeta$ have the same units as the NST quantities $\Omega$ and $\Lambda$: $[\omega] = T^{-1}$, $[\zeta] = LT^{-2}$. Furthermore, $[J] = M/L^2T$, and $[\lambda] = TL^{-1}$, $[\kappa] = 1$. We also have $[\eta] = [M/LT] = ET/L^3$ and $[\bar{\eta}] = [M/L] = [\eta]T$, and $[A] = L/T = [u]$. By using the definition of $F_{\mu\nu}$ given above and by algebraic manipulations, we find the following corollary.

**Corollary 2.** *The tensor expression of the GTF vortex field equations of theorem 7 are given by* $(\mu, \nu = 0, 1, 2, 3)$

$$F^{\mu\nu}_{;\nu} = -F^{\nu\mu}_{;\nu} = \frac{4\pi}{\bar{\eta}} J^\mu, \tag{2}$$

$$\partial^\alpha F^{\beta\mu} + \partial^\mu F^{\alpha\beta} + \partial^\beta F^{\mu\alpha} = 0, \tag{3}$$

By taking the curl and by algebraic manipulations of the GTF equations of table 1, we find the following two corollaries. These two corollaries are directly related to the propagation of the GTF vorticity and swirl. Corollary 3 shows how the NST vorticity $\Omega$ is a source for the GTF vorticity $\omega$, for incompressible fluids, and how the time-rate of change of the current $\partial J/\partial t$ is a source exciting the GTF swirl field. Corollary 4 is obtained by using the definition of $F_{\mu\nu}$ and the result $\Box F_{\mu\nu} = -\frac{4\pi}{\bar{\eta}}\left(\partial_\mu J_\nu - \partial_\nu J_\mu\right)$ to show the 4-vector potential $A$ can be used to determine $\omega$ and $\zeta$.

**Corollary 3.** *The GTF vortex field equations in table 1 imply the following vector wave equations relating the vorticity and swirl fields to the current J and density ρ of the fluid*

$$\left(\frac{\partial^2}{c_m^2 \partial t^2} - \nabla^2\right)\omega = \frac{4\pi}{\bar{\eta}} \nabla \times J, \tag{4}$$

$$\left(\frac{\partial^2}{c_m^2 \partial t^2} - \nabla^2\right)\zeta = -\frac{4\pi}{\bar{\eta}\lambda c_m}\frac{\partial J}{\partial t} + \frac{4\pi c_m}{\lambda\bar{\eta}}\nabla\rho. \tag{5}$$

*Here* $c_m^2 = \kappa\bar{\lambda}/\bar{\kappa}\lambda$.

For incompressible fluids of constant density $\nabla \times J = \rho\Omega$, equation (4) shows the NST vorticity $\Omega$ is the source of the GTF vorticity field $\omega$. For constant density, $\nabla\rho = 0$, so the last term on the right-hand side of equation (5) vanishes. Thus for incompressible fluids, the time rate of change of the NST vorticity $\Omega$ is the source for the GTF swirl field $\zeta$. In this corollary we notice the maximum speed of propagation $c_m$ is determined by the material parameters.

**Corollary 4.** *The swirl* $\zeta$ *and vorticity* $\omega$ *components of the vortex field tensor* $F_{\mu\nu}(\zeta, \omega)$ *can be obtained in terms of a 4-component gauge potential* $(A^\mu) = (\Phi, A^i)$ *which satisfies a vector wave equation* ($\mu = 0, 1, 2, 3$)

$$\left(\frac{\partial^2}{c_m^2 \partial t^2} - \nabla^2\right)A^\mu = \frac{4\pi}{\bar{\eta}} J^\mu. \tag{6}$$

*Here the Lorenz gauge condition* $\partial_\mu A^\mu = 0$ *is used. This equation is often written as* $\Box A^\mu = \frac{4\pi}{\bar{\eta}} J^\mu$, *where* $\Box \equiv \left(\partial_t^2 - \nabla^2\right)$.

The vector potential does not add new degrees of freedom to the theory. In fact, it allows the six linearly independent components of the vortex field of the antisymmetric field tensor $F_{\mu\nu}$ to be computed from four quantities $\left(A_\mu\right)$ simply by differentiation: $F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu)$. From equation (6), one can show the vortex field tensor $F_{\mu\nu}$ also satisfies a wave equation: $\Box F_{\mu\nu} = \frac{4\pi}{\bar{\eta}}\left(\partial_\mu J_\nu - \partial_\nu J_\mu\right)$. Using the definition of $F_{\mu\nu}(\zeta, \omega)$ given in equation (1), allows one to determine the GTF vorticity $\omega$ and swirl $\zeta$ fields. Solution of equation (6) is thus equivalent to the solution of the vortex field equations of table 1.

These corollaries show how the gauge potential 4-vector $\left(A^\mu\right)$ is sourced by the current $\left(J^\mu\right)$ and how $(\zeta, \omega)$ are obtained from the gauge potential and the current 4-vector. The isomorphism between electromagnetic field theory [4] and vortex field theory is readily apparent: the equations for electrodynamical vortex field are of the exact same form as for the fluid dynamical vortex field [3]. Equations (1), providing a formal, linear, invertible (hence 1 to 1) mapping between the electromagnetic field tensor and the fluid dynamical vortex field tensor, can be seen to be a logical consequence of theorem 1. The analogy (isomorphism) also shows the vortex field equations can be expressed as gauge field equations for the vector potential components $A_\mu$ in the same way as in electrodynamics (corollary 4). As shown in corollarys 3 and 4, the theory yields second order wave equations forr which there is an finite limit to the mode propagation speed $c_m$.

Equations (6) can be solved to determine the transverse modes, for instance, for flows in circular, rectangular or helical pipes. The consequences of being able to compute such modes and categorize the modal structure (topology) of fluid flow are far-reaching. A theory of helicity and the topology of *inviscid* or *perfect fluid* flows has already been developed based on an electrodynamic analogy [24–27]. That theory provides a description of helical structures for perfect fluids. Using the present theory, GTF, the limitation to perfect fluids is removed in the context of the NSEs [28, 29]. As a consequence, the work on the topology of perfect fluid flow, e.g., summarized in references [30] and [31], can be derived from GTF thereby providing an extension of the perfect fluid topology theory to a viscous fluid topology theory. Such topological quantities are crucial for understanding and predicting propagation and dissipation of stress-energy in flows with transverse mode structures. Thus there now exists a machinery for computing the vortex field, $F_{\mu\nu}(\zeta, \omega) = \partial_\mu A_\nu - \partial_\mu A_\nu$, just as in electrodynamics. The vortex field, of course, produces stresses and propagates energy in the fluid. These effects must be included in the stress-energy flux balance.

## 3. Deriving the fluid dynamical Lorentz force law and Poynting theorems

In this section, we *derive* the fluid dynamical Lorentz force law and the Poynting theorem. In appendix A we give the detailed definitions of the tensors involved in our discussion. Our formulation of the balance of stress-energy flux is based on the one given in [2].

Tensor analytic methods for a 4D $c_m$-spacetime are used in the remainder of this paper. We follow the conventional notation: subscripts indicate covariant components. Superscripts denote contravariant components. For a vector of covariant components $J_\mu$, the same vector expressed in terms of contravariant components is given by $J^\mu = g^{\mu\nu} J_\nu$, where the Einstein convention of summing over repeated covariant-contravariant index pairs is used. Greek letter indices vary over $\{0, 1, 2, 3\}$. Roman letter ones vary over 1, 2, 3. The metric tensor has components given by $(g^{\mu\nu}) = \mathrm{diag}(1, -1, -1, -1)$ in units where $c_m = 1$, thereby defining the geometry of a Minkowski spacetime. The formulated equations are covariant in form with respect to Lorentz transformations in which $c_m$ is held constant. This is called $c_m$-Lorentz covariance. This covariance follows from the isomorphism of the VFEs and Maxwell's equations, the latter being $c$-Lorentz covariant as the maximum speed of transverse waves is the speed of light $c$. The use of this Minkowski spacetime forms the basis for preserving causal ordering. A $c_m$-covariant theory can be developed on this basis.[3] The covariant ($c_m$-ST) derivative is denoted by a semi-colon and a subscript. For the present discussion these covariant derivatives are equivalent to partial derivatives if cartesian coordinates are adopted. The 4D spacetime tensor approach is both simpler and more elegant compared to the (3+1)D formulation used in table 1 and simplifies the derivations of the Lorentz force law and Poynting's theorem.

### 3.1. Fluid dynamical Lorentz force law

In this and the following, section we attend to the rigorous derivations of the Lorentz force and the Poynting theorem of fluid dynamics. As shown above the fluid dynamical-electrodynamical analogy is not arbitrary. It is a consequence of the fact that both theories describe the dynamics of conserved currents. We start with the balance of stress-energy flux including inertial and vortex field contributions

$$\tau^{\mu\nu}_{e;\,\nu} = -\tau^{\mu\nu}_{m;\,\nu} \equiv F^{\mu\nu} J_\nu. \tag{7}$$

This equation also expresses the balance of stress-energy flux in the fluid. Here the quantity on the left-hand side describes the stress-energy flux due to inertia $\tau_e^{\mu\nu}$. The quantity $\tau_m^{\mu\nu}$ is the stress-energy of the fluid vortex field. For completeness, we give the expressions for these

[3] Physical quantities (scalars, vectors, or tensors) transform $c_m$-covariantly if they transform according to a representation of the Lorentz group having a maximum speed parameter $c_m$. Scalars transform $c_m$-invariantly.

quantities in appendix A, as derived in [2, p 505]. The derivation of the Lorentz force law states the equivalence of this stress to the stress due to the fluid dynamical vortex field. The derivation of the form of the stress exerted by the vortex field amounts to proving the identity $(\tau_m)^{\mu\nu}_{;\nu} \equiv -F^{\mu\nu}J_\nu$. Here the symmetric *Maxwell stress-energy tensor,* also called the *energy-momentum tensor,* tensor $\tau_m^{\mu\nu}$ of the vortex field, is expressed by[4]

$$4\pi\bar{\eta}^{-1}\tau_m^{\mu\nu} = g^{\mu\alpha}F_{\alpha\beta}F^{\beta\nu} + \tfrac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}. \tag{8}$$

As a consequence of the isomorphism between the vortex field equations for electrodynamics and for fluid dynamics, the form of the stress-energy tensor in electrodynamics and fluid dynamics have the same form as given in equation (8). The evaluation of the stress-energy tensor in terms of the quantities on the right of equation (8) is most easily effected by expressing all quantities in terms of their corresponding matrix representations. See also [32, pp 65, 87] and note if $F_{\alpha\beta} = F(\zeta, \omega)$, then $F^{\alpha\beta} = F(-\zeta, \omega)$). We then apply tensor analysis techniques for evaluating the covariant derivative of $\tau_m^{\mu\nu}$ to derive the Lorentz force law as follows. See also [4, section 12.10C]

**Theorem 5.** *Fluid Dynamical Lorentz force law. The force (density) $f^i$ acting on the fluid current due to the fluid vortex field is given by* ($i = 1, 2, 3, \nu = 0, 1, 2, 3$)

$$f^i = F^{i\nu}J_\nu = \left(\lambda\zeta^i J_0 + \kappa(J\times\omega)^i\right), \qquad F^{0\nu}J_\nu = -\lambda\zeta^i J_i. \tag{9}$$

*Here $J^i = \rho u^i$ are the 3-vector components of the fluid current, $\zeta^i$ and $\omega^i$ are the vector and pseudo-vector components of the fluid dynamical vortex field tensor $F^{\mu\nu}$.*

**Proof.** We need to evaluate the covariant derivative $\partial_\nu\left(\tau_m^{\mu\nu}\right) \equiv \tau^{\mu\nu}_{m;\,\nu}$. In this evaluation we use the *identity*

$$\begin{aligned} g^{\mu\alpha}\left(\partial_\nu F_{\alpha\beta}\right)F^{\beta\nu} &= \left(\partial_\nu F^\mu_\beta\right)F^{\beta\nu} = \left(\partial_\nu F^{\mu\alpha}\right)g_{\alpha\beta}F^{\beta\nu},\\ &= \left(\partial_\nu F^{\mu\alpha}\right)F^\nu_\alpha = \left(\partial_\nu F^{\mu\alpha}\right)g^{\nu\beta}F_{\alpha\beta},\\ &= \left(\partial^\beta F^{\mu\alpha}\right)F_{\alpha\beta}. \end{aligned} \tag{10}$$

The fact the covariant derivative of the metric tensor vanishes is used. We also use the *product rule* stating the product of an antisymmetric tensor $F_{\alpha\beta}$ with a symmetric one vanishes, so

$$\tfrac{1}{2}F_{\alpha\beta}\left(\partial^\beta F^{\mu\alpha} + \partial^\alpha F^{\mu\beta}\right) = 0. \tag{11}$$

There follows the sequence of steps leading to the evaluation of the covariant derivative (We set $\bar{\eta}^{-1} = 1$ on both sides of the equation.)

$$4\pi\partial_\nu\left(\tau_m^{\mu\nu}\right) = \partial_\nu\left(g^{\mu\alpha}F_{\alpha\beta}F^{\beta\nu}\right) + \tfrac{1}{4}g^{\mu\nu}\partial_\nu\left(F_{\alpha\beta}F^{\alpha\beta}\right), \tag{12a}$$

[4] See [4, section 12.10B] for a discussion. The right-hand side of equation (8) can also be written as $-g^{\mu\alpha}F^{\mu\sigma}F^\nu_\sigma + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}$. Sign conventions and metrics are often reversed. In [4] and in [2] and [32] which deal with the classical theory of electromagnetic and matter fields, the metric is $(g^{\mu\nu}) = \mathrm{diag}(1, -1, -1, -1)$ in units where $c_m = 1$. In [33, section 56] which deals with classical theory of spacetime fields, the opposite signature is used. Such an overall sign change does not change the field equations, as seen by the fact the (Euler-Lagrange) equations derived from opposite signatures lead to the same results at an extremum [32, section 33]. One can show in both cases the energy-momentum tensor is traceless, $\tau^\nu{}_\nu = 0$.

$$= g^{\mu\alpha}\left(\partial_\nu F_{\alpha\beta}\right)F^{\beta\nu} + g^{\mu\alpha}F_{\alpha\beta}\partial_\nu F^{\beta\nu} + \frac{1}{2}g^{\mu\nu}F_{\alpha\beta}\partial_\nu F^{\alpha\beta}, \tag{12b}$$

$$= g^{\mu\alpha}\left(\partial_\nu F_{\alpha\beta}\right)F^{\beta\nu} - 4\pi g^{\mu\alpha}F_{\alpha\beta}J^\beta + \frac{1}{2}g^{\mu\nu}F_{\alpha\beta}\partial_\nu F^{\alpha\beta}, \tag{12c}$$

$$= -4\pi g^{\mu\alpha}F_{\alpha\beta}J^\beta + \frac{1}{2}F_{\alpha\beta}\left(\partial^\mu F^{\alpha\beta}\right) + g^{\mu\alpha}\left(\partial_\nu F_{\alpha\beta}\right)F^{\beta\nu}, \tag{12d}$$

$$= -4\pi g^{\mu\alpha}F_{\alpha\beta}J^\beta - \frac{1}{2}F_{\alpha\beta}\left(\partial^\beta F^{\mu\alpha} + \partial^\alpha F^{\beta\mu}\right) + F_{\alpha\beta}\partial^\beta F^{\mu\alpha}, \tag{12e}$$

$$= -4\pi F^{\mu\nu}J_\nu + \frac{1}{2}F_{\alpha\beta}\left(\partial^\beta F^{\mu\alpha} + \partial^\alpha F^{\mu\beta}\right), \tag{12f}$$

$$= -4\pi F^{\mu\nu}J_\nu, \tag{12g}$$

$$\partial_\nu\left(\tau_m^{\mu\nu}\right) = -F^{\mu\nu}J_\nu. \tag{12h}$$

Equation (12*a*) expresses the definition of the covariant derivative of $\tau_m^{\mu\nu}$. Evaluating the covariant derivative of the last term of that equation gives equation (12*b*). Equation (12*c*) applies the tensor form of the vortex field equation $\partial_\nu F^{\nu\beta} = 4\pi\bar{\eta}^{-1}J^\beta$ (With $\bar{\eta}^{-1} = 1$.) and corollary 2. The following equation, equation (12*d*), involves a rearrangement. Equation (12*e*) combines the last two terms by applying corollary 2 and the identity, equation (10). To obtain the next equation, equation (12*f*), the relation $g^{\mu\alpha}F_{\alpha\beta}J^\beta = F^{\mu\nu}J_\nu$ is used. To get equation (12*g*) the product rule equation (11), is used. Finally in the last equation, we cancel the factor of $4\pi\bar{\eta}^{-1}$. (With $\bar{\eta}^{-1} = 1$.) Evaluating the Lorentz force term $F^{\mu\nu}J_\nu$ using equation (1), raising indices, which changes the sign of the spatial components in the metric chosen, we obtain

$$\left(F^{\mu\nu}J_\nu\right) = \begin{pmatrix} 0 & -\lambda\zeta_1 & -\lambda\zeta_2 & -\lambda\zeta_3 \\ \lambda\zeta_1 & 0 & -\kappa\omega_3 & \kappa\omega_2 \\ \lambda\zeta_2 & \kappa\omega_3 & 0 & -\kappa\omega_1 \\ \lambda\zeta_3 & -\kappa\omega_2 & \kappa\omega_1 & 0 \end{pmatrix}\begin{pmatrix} J_0 \\ -J_1 \\ -J_2 \\ -J_3 \end{pmatrix} = \begin{pmatrix} \lambda\zeta_1 J_1 + \lambda\zeta_2 J_2 + \lambda\zeta_3 J_3 \\ \lambda\zeta_1 J_0 + \kappa\omega_3 J_2 - \kappa\omega_2 J_3 \\ \lambda\zeta_2 J_0 - \kappa\omega_3 J_1 + \kappa\omega_1 J_3 \\ \lambda\zeta_3 J_0 + \kappa\omega_2 J_1 - \kappa\omega_1 J_2 \end{pmatrix} \tag{13}$$

The theorem statement in equation (9) is obtained from the last three elements of the column vector defined in this equation and from the first element by using the fact $\sum_{i=1}^{3}\lambda\zeta_i J_i = -\lambda\zeta^i J_i$. ■

Examination of the second through fourth elements of the first column of equation (13), shows the right-hand side contains the familiar form of the Lorentz force in the analogy where the $\lambda\zeta^i$ are analogous to the components of the electric field $E^i$ and $J_0$ is analogous to the charge $q$. The second, third, and fourth lines also involve the components of $J \times \omega$. Here, the $\kappa\omega^i$ are analogous to the components of the magnetic induction field $B^i$. So these lines account for the analogy to the $u \times B$ of the electromagnetic Lorentz force law. The second relation of equation (9) is usually omitted in vector analytic derivations. This relation describes effects analogous to the 'resistive' heating caused by the work of the current against the swirl field. It vanishes for vortex fields transverse to the current flow, i.e., when $J \cdot \zeta = 0$. All of the Lorentz force components can be computed, given a fluid dynamical current and constitutive parameters, equations (5).

### *3.2. Fluid dynamical Poynting theorem*

The identity $(\tau_m)^{\mu\nu}_{;\nu} = -F^{\mu\nu}J_\nu$, equation (12*h*), is derived in the proof of the Lorentz force law above. A detailed expression for $\tau_m^{\mu\nu}$ is needed to prove the Poynting theorem. This is provided by evaluating the various tensor quantities in equation (8). The tensor components of $\left(\tau_m^{\mu\nu}\right)$ can be arranged in the form [4, section 12.10B], [32–34]

$$\begin{pmatrix} \tau_m^{00} & \vdots & \tau_m^{01} & \tau_m^{02} & \tau_m^{03} \\ \cdots & \vdots & \cdots & \cdots & \cdots \\ \tau_m^{10} & \vdots & \tau_m^{11} & \tau_m^{12} & \tau_m^{13} \\ \tau_m^{20} & \vdots & \tau_m^{21} & \tau_m^{22} & \tau_m^{23} \\ \tau_m^{30} & \vdots & \tau_m^{31} & \tau_m^{32} & \tau_m^{33} \end{pmatrix} = \frac{\bar{\eta}}{4\pi} \begin{pmatrix} \frac{1}{2}\left(\lambda^2\zeta^2 + \kappa^2\omega^2\right) & \mathrm{col}(\lambda\zeta \times \kappa\omega^j \\ \mathrm{col}(\lambda\zeta \times \kappa\omega)^j & \begin{matrix} -\left(\lambda^2\,\zeta^j\,\zeta^k + \kappa^2\omega^j\,\omega^k\right) \\ +\frac{1}{2}\left(\lambda^2\zeta^2 + \kappa^2\omega^2\right)\delta^{jk} \end{matrix} \end{pmatrix} \tag{14}$$

The lower right-hand block $\tau_m^{jk}$ describes the spatial components of the Maxwell stress-energy tensor. The three spatial components of the top row $\tau_m^{0k}$ or three spatial elements in the left-most column $\tau_m^{k0}$ describe the momentum components of the vortex field stress-energy tensor. The upper left-hand corner gives the energy $\tau_m^{00}$. A relationship between these quantities is determined by differentiating the Maxwell stress-energy tensor. This provides proof of the Poynting theorem that follows.

**Theorem 6.** *(Generalized) Fluid dynamical Poynting theorem. For a simply connected 4D $c_m$-ST manifold and a homogeneous, isotropic fluid having a linear constitutive relation between its vortex field $F(\zeta, \omega)$ and its excitations $H(\bar{\zeta}, \bar{\omega})$, satisfying GTF vortex field equations displayed in table 1, the Poynting relations hold*

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \vec{S} = -\lambda c_m J \cdot \zeta = -\lambda c_m \zeta^i J_i,$$
$$S^i = \left(\frac{c_m \bar{\eta}}{4\pi}\right)(\lambda\zeta \times \kappa\omega)^i, \quad \mathcal{E} = \tfrac{\bar{\eta}}{8\pi}\left(\lambda^2\zeta^2 + \kappa^2\omega^2\right). \tag{15}$$

*Here $\mathcal{E}$ is the fluid vortex field energy, the $S^i$ are the components of the fluid dynamical Poynting vector, $\zeta$ and $\omega$ are the vector components of the fluid vortex field $F^{\mu\nu}$, and $J_i$ are the vector components of the fluid current. The parameters $\kappa$ and $\lambda$ are constitutive parameters for a homogeneous isotropic fluid.*

**Proof.** By using equation (14) in equation (12*h*) and indicating the differentiations, we obtain

$$\frac{\partial}{\partial x^\nu} \begin{pmatrix} \frac{\bar{\eta}}{8\pi}\left(\lambda^2\zeta^2 + \kappa^2\omega^2\right)\delta^{0\nu} & \frac{\bar{\eta}}{4\pi}(\lambda\zeta \times \kappa\omega)\delta^{j\nu} \\ \frac{\bar{\eta}}{4\pi}(\lambda\zeta \times \kappa\omega)\delta^{j\nu} & -\frac{\bar{\eta}}{4\pi}\left(\lambda^2\,\zeta^j\,\zeta^k + \kappa^2\omega^j\omega^k + \frac{\bar{\eta}}{8\pi}\left(\lambda^2\zeta^2 + \kappa^2\omega^2\right)\delta^{jk}\delta^{j\nu}\right) \end{pmatrix}$$
$$= \begin{pmatrix} -\lambda\zeta^i J_i \\ -\left(\lambda\zeta^j J_0 + \kappa\,(J \times \omega)^j\right) \end{pmatrix}. \tag{16}$$

Using $x^0 = c_m t$, we then obtain the Poynting theorem of energy density flux from the time-components:

$$\frac{\partial}{\partial c_m t}\left(\tfrac{\bar{\eta}}{8\pi}\left(\lambda^2\zeta^2 + \kappa^2\omega^2\right)\right) + \frac{\partial\left(\bar{\eta}\lambda\zeta \times \kappa\omega\right)^i}{4\pi\partial x^i} = -\lambda\zeta^i J_i, \tag{17}$$

and also a relation for momentum or stress flux (having divided the equation by $c_m$) from the spatial components:

$$\frac{\partial}{\partial x^k}\left(\frac{\bar{\eta}}{8\pi}(2\lambda\zeta\times\kappa\omega)^k+\left(\lambda^2\zeta^2+\kappa^2\omega^2\right)\delta^{jk}\right)-\frac{\bar{\eta}}{4\pi}\left(\lambda^2\,\zeta^j\,\zeta^k+\kappa^2\omega^j\omega^k\right)$$
$$=-\left(\lambda\zeta^jJ_0+\kappa\,(J\times\omega)^j\right). \quad (18)$$

This equation provides a generalization of the standard Poynting theorem which is usually limited to equation (17). On comparing equations (15) and (17), the Poynting theorem, equation (15), follows.

Examining the structure and physical units used in equations (15) through (18) shows the vortex field modes transport energy and momentum or equivalently stress-energy. More specifically, since $[\omega]=T^{-1}$, $[\zeta]=LT^{-2}$, $[\lambda]=TL^{-1}$, $[\kappa]=1$, $[\bar{\eta}]=[M/L]$, $[\eta]=[M/LT]$, and $[J]=ET^2L^{-2}L^{-3}$, the terms in Poynting theorem, equation (17), are dimensionally homogeneous with dimension $\frac{E}{L^3}\times\frac{1}{T}$; energy dissipation rate/unit volume. The dissipation due to $-\lambda c_m J\cdot\zeta$ is seen to represent an energy-dissipation rate/unit volume caused by currents working against the swirl field. The term $-\lambda c_m J\cdot\zeta$ gives a velocity dependent contribution to the energy-momentum dissipation, thereby lending stability to the flow equations in the otherwise unstable high Re limit [35], [36]. This energy-dissipation is analogous to a $V\cdot I$ resistive or an $I^2R$ loss in electrodynamics leading to Joule heating. The vector $\vec{S}=\frac{c_m\bar{\eta}}{4\pi}(\lambda\zeta\times\kappa\omega)$ is the analog of the Poynting vector in electromagnetic theory, so this term describes the net momentum flux $\nabla\cdot\vec{S}/c_m$ of the mode system. Let us define an energy-momentum density 4-vector $S=(\mathcal{E},\,S^x/c_m,\,S^y/c_mS^z/c_m)$. Here we include $c_m$ in the metric. Therefore, the fluid dynamical Poynting theorem given in equation (17) can be interpreted physically as stating the sum of the rate of change of the energy-momentum density $\partial_\mu S^\mu$ of the transverse mode excitations is limited to the rate $\lambda\zeta^iJ_i$ at which the flow can generate heat, $\partial_\mu S^\mu=-\lambda\zeta^iJ_i$. The transport of stress-energy stated in the Poynting theorem, equation (17), describes *new, propagating*, channels of energy transport absent in the NST.

## 4. Implications of the Lorentz force law and Poynting theorem

### *4.1. Field equations reflecting new channels of stress-energy transport*

The Lorentz force law and Poynting theorem describe the physical effects incorporated into the GTF, a $c_m$-Lorentz covariant theory of fluid flow. This theory comprises the the equations expressing the conservation of the sum of the stress-energy of the inertia $\tau_e^{\mu\nu}$ of the fluid plus the stress-energy of the vortex field $\tau_m^{\mu\nu}$, the equations of the fluid vortex field, and the definitions of the current $J^\mu$ and vector potential $A^\mu$:

$$\left(\tau_e^{\mu\nu}+\tau_m^{\mu\nu}\right)_{;\nu}=0.$$
$$\left(\tau_e^{\mu\nu}\right)_{;\nu}=-\tau_{m;\,\nu}^{\mu\nu}\equiv F^{\mu\nu}J_\nu,$$
$$J^\mu\equiv(4\pi)^{-1}\bar{\eta}F^{\mu\nu}{}_{;\nu}\,,\qquad \Box A^\mu=\frac{4\pi}{\bar{\eta}}J^\mu,$$
$$F_{\mu\nu}=\left(\partial_\mu A_\nu-\partial_\nu A_\mu\right). \quad (19)$$

The first equation can also be interpreted as stating the balance of inertial stress-energy flux and fluid vortex field flux. In the second equation, the right-hand side is evaluated by using theorem 5 showing the fluid is driven by the Lorentz force. This balances with the inertia of the fluid on the left-hand side. Included in the latter is the pressure of the fluid. On the left-hand side of the second equation, the convective behavior of the motion of matter is expressed. On the right-hand side, the propagation of the vortex field working against the current is expressed. The Poynting theorem theorem 6 gives the details, showing the fact the flow can be dissipative $-\lambda c_m \zeta^i J_i \neq 0$. The third equation gives a covariant definition of the current 4-vector $J^\mu$, in contradistinction to the components of the second order stress-energy tensor described in [2] as explained in appendix A. In appendix B we show $\bar{\eta} = c_m/\bar{\lambda}$. The wave equations determining the vortex field are hyperbolic and causal. The first equation is a constraint that energy-momentum be conserved. The other equations are definitions facilitating the computation of the current and the vortex field. The equations are not restricted to incompressible fluids. The introduction of the 4-vector potential $A^\mu$ introduces a scalar potential $\Phi$, $\left(A^\mu\right) = (\Phi, A^1, A^2, A^3)$. This potential is in a sense a velocity potential: its gradient gives a force acting on unit density elements of the fluid. The current $J^\mu \equiv (4\pi)^{-1}\bar{\eta}F^{\mu\nu}{}_{;\nu}$ is a $c_m$-Lorentz covariant 4-vector defined by the fluid vortex field.

### *4.2. Balance of stress-energy in high speed limits*

In this section we examine the conundrum of the disappearance of viscous stress effects at high Reynolds numbers. In our analysis we first combine the newtonian viscous stresses in the stress-energy flux balance equations togetherr with the stress-energy of the vortex field. (An analysis of the validity of this combination is provided in the next section.) In this manner we can examine the relative scale of terms for time-dependent flow as the Reynolds number $\mathrm{Re} = \parallel u \parallel L_0/(\eta/\rho) \equiv U_0 L_0/\nu$ increases. The combination gives the following *stress-energy flux balance equation* [3] ($\mu, \nu = 0, 1, 2, 3$)

$$\left(\tau_e^{\mu\nu} + \tau_n^{\mu\nu}\right)_{;\nu} = -\tau_{m;\,\nu}^{\mu\nu} \equiv F^{\mu\nu}J_\nu. \tag{20}$$

Here $\tau_e^{\mu\nu}$ is the stress-energy tensor of inertia, $\tau_n^{\mu\nu}$ is the stress-energy tensor due to the newtonian viscous stresses and $\tau_m^{\mu\nu}$ is the (Maxwell) stress-energy tensor of the fluid vortex field [3]. The detailed expressions for these tensors is given in appendix A and a dimensionally analyzed version of the equation is given below as equation (22). For the moment we focus on the structure of the equations. The energy and momentum (equivalently stress-energy) are coupled in equation (20), reflecting the coupling of space and time into one geometric structure. The last equality in equation (20) is a mathematical identity obtained by evaluating the covariant derivative of $\tau_m^{\mu\nu}$ as described below. The identity in the last part of equation (20) provides the basis for the derivation of the fluid dynamical Poynting theorem. In fact, the last term contains the Lorentz force which is seen to be in balance with the inertial and viscous force flux.

We next analyze the structure of equation (20) by expressing it in dimensionless form. The density $\rho$ of the fluid is assumed constant for simplicity. We proceed to use the definitions given in appendix A to evaluate the left-hand side of equation (20) for a cartesian coordinate system giving:

$$\begin{aligned}
(\tau_e + \tau_n)^{\mu\nu}_{,\nu} &= \rho\left(u^\nu \frac{\partial u^\mu}{\partial x^\nu} + u^\mu \frac{\partial u^\nu}{\partial x^\nu}\right) + \frac{\partial p}{\partial x^\nu}\mathcal{P}^{\mu\nu} - \eta\frac{\partial^2 u^\mu}{\partial x^\nu \partial x^\varepsilon}\mathcal{P}^{\varepsilon\nu}, \\
&= \rho\left(u^\nu \frac{\partial u^\mu}{\partial x^\nu} + u^\mu \frac{\partial u^\nu}{\partial x^\nu}\right) + \frac{\partial p}{\partial x^\nu}\delta^{\mu\nu} - \frac{\eta}{2}\left(\frac{\partial^2 u^\mu}{\partial x^\nu \partial x^\varepsilon}\mathcal{P}^{\varepsilon\nu} + \frac{\partial^2 u^\mu}{\partial x^\varepsilon \partial x^\nu}\mathcal{P}^{\nu\varepsilon}\right) \\
&= \rho u^\nu \frac{\partial u^\mu}{\partial x^\nu} + \frac{\partial p}{\partial x^\nu}\delta^{\mu\nu} - \eta\frac{\partial \tilde{\sigma}^{\mu\nu}}{\partial x^\nu}.
\end{aligned} \tag{21}$$

These equations are limited to the case $\| u \| \ll c_m$. Here the spatial projection operators $\mathcal{P}^{\mu\nu}$ are defined in appendix A. This result can be non-dimensionalized (as indicated by the $\star$-subscripts and the scale factors $U_0$, $L_0$, $P_0$, and $T_0$), simplified by dropping quadratic terms in $u^\mu$ such as $\left(-U_0^2/c_m^2\right) u_\star^\mu u_\star^\nu \left(\partial p_\star/\partial x_\star^\nu\right)$, as they would be small when $\| u \|/c_m = U_0/c_m \ll 1$, then combining terms with the right-hand side of equation (20) giving

$$u_\star^\nu \frac{\partial u_\star^\mu}{\partial x_\star^\nu} + \frac{1}{\rho U_0^2/P_0}\left(\frac{\partial p_\star}{\partial x_\star^\nu}\delta^{\mu\nu}\right) = \frac{\eta}{\rho U_0 L_0}\frac{\partial \tilde{\sigma}_\star^{\mu\nu}}{\partial x_\star^\nu} + \frac{L_0}{T_0}\frac{1}{U_0}F_\star^{\mu\nu}J_{\star\nu},$$
$$= \mathrm{Re}^{-1}\frac{\partial \tilde{\sigma}_\star^{\mu\nu}}{\partial x_\star^\nu} + \frac{1}{U_0/\left(L_0/T_0\right)}F_\star^{\mu\nu}J_{\star\nu}. \tag{22}$$

The fluid pressure is given by $p$ and the Cauchy stress tensor, generalized for a 4D $c_m$-spacetime is denoted by $\tilde{\sigma}^{\mu\nu}$. (See appendix A.) Because of the inclusion of the newtonian viscous stress these equations are acausal: we notice some immediate parallels to the non-dimensionalized NSEs. The first term on the left contains the spatial and temporal gradients comprising the total derivative $Du^i/Dt$ of the NSEs. The pressure gradient is also present on the left-hand side. Since we have included a newtonian viscosity, the spatial derivatives of the Cauchy strain rate tensor $\tilde{\sigma}_\star^{\mu\nu}$ also appears, on the right-hand side. These terms, however, are augmented by temporal derivatives. The new term on the far right contains the effects due to the fluid vortex field. Equation (22) contains the Navier-Stokes equations with a Reynolds number dependence ($\mathrm{Re}^{-1} = [\eta/\rho]/\left[U_0 L_0\right]$) as well as a new forcing term having a new dimensionless group which is of the form of the ratio of two characteristic velocities $\left(L_0/T_0\right)U_0^{-1}$. The $T_0$ comes from the vortex field and $L_0$ from the geometric scale of the flow tube. The ratio is therefore a vortex field speed to the fluid speed. From equation (22) it is clear that the dimensionless group $\left(L_0/T_0\right)U_0^{-1}$ multiplying the non-dimensionalized vortex field effects $F_\star^{\mu\nu}J_{\star\nu}$ is independent of the Reynolds number.

We can examine the implications of the new dimensionless grouping in equation (22) as follows. Consider a fixed characteristic value of the ratio of the dimensionless group of velocities $\left(L_0/T_0\right)U_0^{-1}$. The limit of negligible viscous stresses is then found by allowing the Reynolds number Re to approach infinity (e.g., $\nu = \eta/\rho \to 0$) in a way that leaves the last term in equation (22) non-vanishing but scaled by the fixed ratio of velocities $\left(L_0/T_0\right)U_0^{-1}$. This yields a high Re limit where inertial and vortex field stresses dominate with the viscous stresses contributing little. Recall at high Re the NSEs describe just the inertial forces as if the fluid were a perfect fluid. The new term, the Lorentz force term $F_\star^{\mu\nu}J_\nu^*$, does not depend directly on the Reynolds number (newtonian viscosity) and remains even as $\mathrm{Re} \to \infty$. Equation (22) shows viscous stresses are decoupled from the vortex field dynamical effects (proportional to $F_\star^{\mu\nu}J_\nu^*$). The remaining part of the equations describe the causal balance of inertial and vortex field stress-energies. Therefore using the GTF, it is possible to formulate a causal theory of time-dependent flow as $\mathrm{Re} \to \infty$, i.e., in the limit of vanishing newtonian viscous stresses. That is, as the newtonian viscous stress contribution to the stress-energy flux balance vanishes, the acausality due to newtonian viscosity is removed. In the $\mathrm{Re} \to \infty$ limit, the forces determining the current remain in balance: that balance being struck by the inertial and the vortex field stress-energies. In such a theory, new channels of energy dissipation and transport due to the vortex field modes are active.

*4.3. Characterization of time-independent and time-dependent flows*

An analysis of how to include the vortex field and newtonian viscous stresses into the stress-energy flux balance equations is presented in this section. For simplicity, we consider the constant density, small flow speed limit (relative to the maximum transverse mode speed $c_m$, i.e., $\| u \|/c_m \ll 1$) of the stress–energy flux balance. This limit yields the following set of equations [2, 37]

$$u \cdot \nabla u^i + \frac{1}{\rho} \nabla^i p = \nu \nabla^2 u^i \tag{23a}$$

$$\frac{\partial u^i}{\partial t} + u \cdot \nabla u^i + \frac{1}{\rho} \nabla^i p = \lambda \zeta^i + \frac{\kappa}{c_m} (u \times \omega)^i, \tag{23b}$$

$$\begin{aligned} &\nabla \cdot \omega = 0 \qquad \nabla \times \zeta + \frac{\partial \omega}{\partial t} = 0 \\ &\nabla \cdot \bar{\zeta} = 4\pi\rho \qquad -\frac{1}{c_m}\frac{\partial \bar{\zeta}}{\partial t} + \nabla \times \varpi = \frac{4\pi}{c_m} J \end{aligned} \tag{23c}$$

As displayed, equation (23*a*) contains the Navier-Stokes term $(\eta/\rho) \nabla^2 u^i \equiv \nu \nabla^2 u^i$ due to newtonian viscosity for time-independent flow. This term provides the balance to the inertial stresses when there is no concern about acausal action-at-a-distance (newtonian physics). The equation is mathematically classified as an elliptic boundary-value problem. One can use equation (23*a*) to compute the steady, laminar flow. This laminar time-independent flow equation can be considered to provide a ‘vacuum’ or reference state from which time-dependent flow emerges as, say, pressure drop is increased. In short, equation (23*a*) is is formulated in a context where causality does not enter—because no time-dependence is involved in its formulation. The results of equation (23*a*) can be obtained, by solving for the time-independent limit of equations (23*b*-23*c*), i.e., for stationary fields. In this case the energy-dissipation is still given by the Poynting theorem as $\lambda J \cdot \zeta$. Equation (23*b*) for time-dependent flows contains the effects of the Lorentz force $f^i/\rho = F^{i\nu} J_\nu = \lambda \zeta^i + \kappa/c_m (u \times \omega)^i$ exciting the fluid and replacing the newtonian viscous stress term. Although equation (23*b*) is mathematically the large Reynolds number limit of equation (22) above, equation (23*b*) should not to be considered as a high Re limit of a time-dependent NST; the equation stands on its own as describing a time-dependent dissipative fluid flow. For a given geometry, the pressure or flow rate, at which the transition to time-dependent flow occurs, i.e., where the solutions to equation (23*a*) transfer to those of equation (23*b*), can be determined by simultaneously solving equations (23*b*-23*c*). When the time-dependent vortex modes are excited, a substantial increase in energy dissipation occurs and can be measured [39]. Equations (23*a*-23*c*) are thus hybrid equations for laminar and time-dependent flows. These equations include as a stationary case a formulation in terms of the effects of newtonian viscous stresses. For the dynamical case and its stationary limit, the equations containss the effects of the vortex field.

The physical picture is that the vortex field modes are added (subtracted) to (from) the vacuum state as higher levels of excitation occur. The vortex field modes or elementary excitations are solutions to the vortex field equations (equation (23*c*)). These can be used as a basis set to define the elementary excitations. The vortex field modes can be obtained using the wave equation for the vector potential—See corollary 4. The wave operator is essentially self-adjoint, generating a complete set of basis functions. The propagation of stress-energy by such modes can impact the linear stability and stress-energy transport (convective and propagational) analysis.

## 5. Summary and conclusions

This paper provides an elaboration of our theoretical description of fluid flow called the geometrodynamical theory of fluids (GTF). The paper is focused on two key ingredients of that theory: the Lorentz force Law and the Poynting theorem. The GTF, itself, is based on the mathematical theory of conserved currents as reflected in the vortex field lemma. Exploiting the lemma allows introducing a 4-vector current definition, a tensor formulation, and the introduction of causal equations to describe fluid dynamics. The result is a causal, covariant field theory of fluid flow. A remarkable aspect of the GTF is it contains a subset of equations, for what we call the fluid vortex field, that are mathematically isomorphic to the Maxwell's equations of electrodynamics. Consequently, we are able to derive a fluid dynamical Lorentz force law and fluid dynamical Poynting theorem following approaches established in the theory of electrodynamics. This provides a basis for a fluid dynamical-electrodynamical analogy—the theory shows the GTF equations and those of electrodynamics (EMT) are isomorphic. We show the fluid dynamical Lorentz force is produced by the fluid vortex field generated by the conserved fluid current. The fluid vortex field, in turn, modifies the stress-energy flux balance of the fluid. This leads to new causal channels of stress-energy transport and dissipation which are described by the fluid Poynting theorem. These channels persist in the high Reynolds number limit ($\mathrm{Re} \to \infty$) providing a balancing stress to the inertial stresses in the fluid and also providing stress-energy dissipation. These effects are absent from the NST. Because the GTF is causal and covariant—a modern field theory—it is likely that it can form part of a successful theory of turbulence.

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## Appendix A. Covariant current, Maxwell, Euler, and Navier-Stokes stress-energy tensors

The definitions of the covariant 4-current, Maxwell, Euler, and Navier-Stokes stress-energy tensors are given in a $c_m$-Lorentz covariant form compatible with the VFEs. The maximum speed of transverse waves is $c_m$, which we set to unity except for emphasis.

For the following discussion of the 4-vector nature of the current $J$, we need the vortex field lemma (VFL) giving the field equations relating the current density $^{\star}J$ to the fluid excitations $H$. The VFL is a fundamental consequence of the constraint of fluid conservation that is obeyed for all classical theories of continua [3]. The lemma follows on using the principle of current density conservation [3] and the converse of the Poincaré lemma stating an closed differential form $F$, i.e., one for which $dF = 0$, locally has a potential $f = d\alpha$ [38, p 27].

**Lemma 7.** *Vortex Field Lemma (VFL, Scofield and Huq, [3]). For a contractible 4D spacetime manifold with conserved currents, ${}^{\star}J$-conservation (equivalently, the continuity equation) implies the vortex field equations*

$$dH = 4\pi^{\star}J, \tag{A.1}$$

$$dF = 0. \tag{A.2}$$

*Here H is the vortex field excitation 2-form, ${}^{\star}J$ is the current density 3-form, $F = dA$ is the gauge degree of freedom of H (i.e., $dH = d(H + F)$), called the vortex field strength 2-form, and A is its gauge potential, a 1-form.*

**Theorem 8.** *Conserved 4-Vector Current. Given the constitutive relations $H =^{\star} F$, a Lorentz covariant current is derived from the vortex field by $J^{\mu} \equiv (4\pi)^{-1}\bar{\eta}F^{\mu\nu}{}_{;\nu}$. This current is necessarily conserved.*

**proof.** By the vortex field lemma $d^{\star}H = 4\pi^{\star}J$ implies $d^{\star}J = 0$ and consequently the continuity equation, $\partial_{\mu}J^{\mu} = 0$, and conversely [3]. From equation A.1), we have ${}^{\star}J = (4\pi)^{-1}dH$. Substituting from the constitutive relations into this equation, we have ${}^{\star}J = (4\pi)^{-1}d^{\star}F$. Taking the Hodge-$\star$ of this equation for 4D spacetimes with signature 2 gives the definition of the current 1-form: $J = (4\pi)^{-1\star}d^{\star}F = (4\pi)^{-1}\ {}^{\star}d^{\star}F = (4\pi)^{-1}\ \delta F$, where we have introduced the codifferential $\delta = {}^{\star}d^{\star}$. This result is equivalent to the tensor definition $J^{\mu} \equiv (4\pi)^{-1}\bar{\eta}F^{\mu\nu}{}_{;\nu}$, where the factor of $\bar{\eta}$ has been reintroduced to make the tensor relation consistent with the units chosen. Using the constitutive relations $H = {}^{\star}F$, we can take the covariant definition of current to be

$$J^{\mu} \equiv (4\pi)^{-1}\bar{\eta}F^{\mu\nu}{}_{;\nu}\,. \tag{A.3}$$

Since $F^{\mu\nu}$ is covariant (in fact Lorentz covariant, based on the isomorphism of Maxwell's equations and the VFEs; Maxwell's equations are $c$ -Lorentz covariant) and covariant derivatives are also covariant. Since ${}^{\star}J = (4\pi)^{-1}d^{\star}F$, $d^{\star}J = (4\pi)^{-1}d^{2\star}F = 0$, the current $J^{\mu}$ defined this way is a necessarily a conserved 4-vector.

The current must be self-consistently computed from the vortex field equations. The solutions to the vortex field equations can be constrained, for instance, by requiring them to satisfy a stress-energy flux balance as described in the main body of the paper.

This definition and the developments in the main body emphasize the fact the current $J$ is the physical quantity. Only when the density is a constant can we write $\left(J^{\mu}\right) = \left(\rho_0 c_m, \rho_0 u^1, \rho_0 u^2 \rho_0 u^3\right)$. In this case $\rho_0$ is assumed to be a Lorentz scalar. Since the velocity is Lorentz covariant, this expression for the current is Lorentz covariant. If $\rho$ is not a constant, since $c_m$ is a constant then we can define $\left(J^{\mu}\right) \equiv \left(\rho c_m, \rho u^1, \rho u^2 \rho u^3\right)$ by setting $u^{\mu} = J^{\mu}/c_m$. The Lorentz force 1-form can be constructed as an interior product (or contraction):

$$i_J\left(F_{\mu\nu}dx^{\mu} \wedge dx^{\nu}\right) = F_{\mu\nu}J^{\nu}dx^{\mu}. \tag{A.4}$$

**Definition 9.** Maxwell stress tensor $\left(\tau_m^{\mu\nu}\right)$. See equation (14). Our convention for units implies $\kappa$ is unitless.

**Definition 10.** Euler inertial stress-energy tensor. The fluid inertia stress-energy (density) tensor $\tau_{\rm e}$ is given by [2, Ch XV], [10]

$$\tau_e^{\mu\nu} = \rho u^\mu u^\nu + p\left(g^{\mu\nu} + c_m^{-2} u^\mu u^\nu\right) = h u^\mu u^\nu + p g^{\mu\nu}. \tag{A.5}$$

Here $u^\mu$ is the 4-velocity a first order tensor. The quantity $h = \varepsilon + p c_m^{-2}$ is the heat function per unit volume (enthalpy) of the fluid; $\varepsilon$ is the internal energy per unit volume (including rest mass-energy $\rho c^2$), and $p$ is the pressure referred to proper ($c_m$-Lorentz covariant) volumes in energy units. The enthalpy for low speed $\| u \| / c_m \ll 1$, incompressible flows is essentially the constant mass-density. The stress-energy density tensor in $4D$ $c_m$-spacetime ($c_m$-ST) on inserting physical components of velocity, $u$, is

$$\tau_e^{00} = \frac{h}{1-\beta^2} - p = \frac{\varepsilon + p\beta^2}{1-\beta^2},\ \tau_e^{\alpha 0} = -\frac{h v^\alpha}{c_m\left(1-\beta^2\right)},$$
$$\tau_e^{\alpha\gamma} = \frac{h v^\alpha v^\gamma}{c_m^2\left(1-\beta^2\right)} + p\delta^{\alpha\gamma}, \quad \beta = v/c_m. \tag{A.6}$$

The metric of the $c_m$-ST is $\left(g_{\mu\nu}\right) = \mathrm{diag}(1, -1, -1, -1)$, $c_m = 1$ and the normalized velocity vector components $\hat{u}_\mu$. The spatial current components in the limit $\| u \| \ll c_m$ are approximately equal to $\rho u + \mathcal{O}\left(u/c_m^2\right)$, thus limit to $\rho u$. The temporal part of the energy-momentum tensor $\tau_e^{00}$ limits to $\rho c_m^2 + \rho\varepsilon + (1/2)\rho u^2$, so the momentum $\tau_e^{00}/c_m$ limits to $\rho c_m$ [32, p 506]. Thus, the limits of the temporal components of the energy-momentum second order tensor $\tau_e^{\alpha 0}/c_m$ also give a current. The quantity $\tau_e^{\mu\nu}$ is covariant in the sense it is a tensor, unchanged in form under differential coordinate transformations, specifically, the equations of motion are $c_m$-Lorentz covariant. For larger velocities compared to $c_m$ the full apparatus of tensor analysis must be used.

This current is formed from elements that are the time components $\left(\tau_e^{00}, \tau_e^{10}, \tau_e^{20}, \tau_e^{30}\right)$ of the second-order energy-momentum tensor $\tau_e^{\mu\nu}$. In the most general case these quantities do not form components of a 4-vector. Thus, the covariant 4-vector definition of the necessarily conserved current vector of theorem 8 is required. In the case of constant density $\rho_0$, the fluid analog current vector is invariant with respect to spacetime (Lorentz) transformations because the velocity 4-vector is Lorentz transformation invariant in the same way as in electrodynamics. At small enough flow speeds $\| u \| \ll c_m$ these considerations are of small practical importance for either electrodynamics or fluid dynamics.

**Definition 11.** Navier-Stokes viscous stress-energy tensor. The Navier-Stokes stress-energy *(density)* tensor is given by [2, *Ch. XV*], [33, section 22.3]

$$\tau_{\mu\nu}^n = -2\eta\left(\tilde{\sigma}_{\mu\nu} + \delta\theta\mathcal{P}_{\mu\nu}\right). \tag{A.7}$$

The signature convention is such that stress-energies at a point appear with plus signs when summed at a point. Here $\tilde{\sigma}_{\mu\nu} = \frac{1}{2}\left(\mathcal{P}^\epsilon_{\ \nu} u_{\mu;\epsilon} + \mathcal{P}^\epsilon_{\ \mu} u_{\nu;\epsilon}\right) - \frac{1}{3}\theta\mathcal{P}_{\mu\nu}$, where the projection operator $\mathcal{P}_{\mu\nu}$ to a spatial 3-volume perpendicular to the spacetime 4-vector $\hat{u}$ ($\hat{u}\cdot\hat{u} = 1$) is $\mathcal{P}_{\mu\nu} = \delta_{\mu\nu} - \hat{u}_\mu\hat{u}_\nu$. The operators $Q_{\mu\nu} = \delta_{\mu\nu} - \mathcal{P}_{\mu\nu}$ project vector components onto the fluid pathlines (world lines in the spacetime). The quantity $\theta = u^\mu_{;\mu}$, $\eta$ is the absolute viscosity, and $\delta$ is a dilatation viscosity effect coefficient arising from dissipation due to compression/expansion of the fluid. The conditions

$$u_{\mu u}\tau_n^{\mu\nu} = 0, \quad \tau_\nu^\nu = 0, \tag{A.8}$$

serve to restrict the structure of the relativistic extension of the stress tensor.

Choosing the galilean time axis to be aligned with the proper time axis at a point and using a cartesian coordinate system gives $(u^\nu) = (1, 0, 0, 0)$ and $(u_\nu) = (1, 0, 0, 0)$ in the moving frame of the fluid so $\left(\mathcal{P}_{\mu\nu}\right) = \left(\mathcal{P}^{\mu\nu}\right) = \mathrm{diag}(0, 1, 1, 1)$. So, the 4-velocities $u_\mu$ have the normalization $u^\mu u_\mu = 1$, $c_m = 1$. In this frame $\tau_n^{00} = \tau_n^{0\nu} = \tau_n^{\mu 0} = 0$ and for an incompressible fluid

$$\tau_{ij} \to \eta\left(\frac{\partial u^i}{\partial x^i} + \frac{\partial u^i}{\partial x^i}\right), \tag{A.9}$$

as required. We also have

$$\frac{\partial \tau_n^{\mu\nu}}{\partial x^\nu} = \eta\left(\frac{\partial^2 u^\mu}{\partial x^\nu \partial x^\varepsilon}\mathcal{P}^{\varepsilon\nu} + \frac{\partial^2 u^\nu}{\partial x^\nu \partial x^\varepsilon}\mathcal{P}^{\varepsilon\mu}\right) \to \eta\frac{\partial^2 u^\mu}{\partial x^\nu \partial x^\varepsilon}\mathcal{P}^{\varepsilon\nu} = \eta\nabla^2 u^\mu. \tag{A.10}$$

The last equation holds for an incompressible fluid and a cartesian coordinate system. Terms involving $u^\mu u^\nu / c_m^2$ in the projectors $\mathcal{P}^{\varepsilon\nu}$ to the 3D spatial manifold can be neglected under the assumption $u^\mu u^\nu / c_m^2 \ll 1$.

## Appendix B. Constitutive parameter set reduction

The stress-energy flux equations of [2] based on the inertial and viscous stresses omit the full consequences of the conservation of current embodied by the vortex field equations. The viscous stress terms are Lorentz covariant but lead to the problem of acausality as discussed in the main text. The acausality problem is removed when the GTF vortex field is included.

It is well known that the number of (linear response) independent constitutive coefficients $C_{\kappa\lambda}^{\mu\nu}$ for homogeneous, isotropic fluids is only two. This implies from the quantities $\left\{\bar{\lambda}, \bar{\kappa}, \lambda, \kappa\right\}$ appearing in table 1 that only two, the ratios of these quantities, namely $\left\{\bar{\lambda}/\lambda, \kappa/\bar{\kappa}\right\}$, are independent. In the main text, we also note the total number of constitutive parameters in the set $\{\mu, \kappa, \bar{\mu}, \bar{\kappa}, \bar{\eta}, c_m\}$ can be reduced to three. This is a convenience, and has some merit of economy from both a theoretical and an experimental standpoint. From the experimental standpoint, there are fewer parameters to measure. From the theoretical standpoint there are fewer parameters to compute. The problem in the present case is to make the best choice of parameters. In electrodynamics, the arrangement is not entirely satisfactory as the fields and excitations are related by $(D, H) = (\varepsilon E, \mu^{-1}B)$ involving an inverse. Thus, in this instance, electrodynamics is not the best guide. Instead, here we minimize the number of constitutive parameters in a way that the units of $\omega$ and $\zeta$ are those of the corresponding NST quantities $\Omega = \nabla \times u$ and $\Lambda = \Omega \times u$, namely, $[\omega] = T^{-1}$, $[\zeta] = LT^{-2}$ for incompressible fluids. We show the parameter $\bar{\eta}$ can be removed as in electrodynamics. As we have seen in the main text, choosing the units of $\omega$ and $\zeta$ to have the same units as the NST quantities $\Omega$ and $\Lambda$, simplifies the interpretation of the theory. On the other hand from a theoretical standpoint, the full set of parameters reveals symmetries and derivation pathways in the theory that a rigid parametrization does not. We have used this approach in analyzing classical field theories [3].

We first derive the VFEs appearing in lemma 7 then apply dimensional analysis yielding an exact formal map of the fluid to the electrodynamic VFEs in the form of Maxwell's equations. This permits us to show how the assumption of homogeneity and isotropy enter the theory. The vortex field lemma (VFL, Lemma 7) shows the continuity equation, $\partial_\mu J^\mu = 0$, is equivalent to the differential geometric relations $dH = 4\pi {}^*J/\bar{\eta}$ and $dF = 0$. There exists a 1-form $A$ such that $F = dA$ because of the Poincaré lemma. In the language of the exterior calculus, the constitutive relations are given by $-{}^*H = F$. Using these results we can derive the simplified VFEs corresponding to lemma 7 as follows.

It is first recalled, in the exterior calculus, the 'wedge' product is defined as $\alpha \wedge \beta = -\beta \wedge \alpha$. The exterior derivative $d$ of a wedge product of $q$ and $p$- forms is given by the relation $d\left(\overset{q}{\alpha} \wedge \overset{p}{\alpha}\right) = d\overset{q}{\alpha} \wedge \overset{p}{\alpha} + (-1)^q \overset{q}{\alpha} \wedge d\overset{p}{\alpha}$ [38, p. 20]. The exterior derivative of the $p$-form $\overset{p}{\omega}$ satisfies Poincaré's lemma $d^2\overset{p}{\omega} = 0$. In terms of this calculus, the fluid excitation 2-form $H$ and the vortex field 2-form $F$ can be written as

$$\begin{aligned} -{}^*H &= \bar{\lambda}\left(\bar{\zeta}_1 dx^1 + \bar{\zeta}_2 dx^2 + \bar{\zeta}_3 dx^3\right) \wedge c_m dx^0 \\ &\quad + \bar{\kappa}\left(\bar{\omega}_1 dx^2 \wedge dx^3 + \bar{\omega}_2 dx^3 \wedge dx^1 + \bar{\omega}_3 dx^1 \wedge dx^2\right), \\ F &= \lambda\left(\zeta_1 dx^1 + \zeta_2 dx^2 + \zeta_3 dx^3\right) \wedge c_m dx^0 \\ &\quad + \kappa\left(\omega_1 dx^2 \wedge dx^3 + \omega_2 dx^3 \wedge dx^1 + \omega_3 dx^1 \wedge dx^2\right). \end{aligned} \tag{B.1}$$

The constitutive relation $-{}^*H = F$, by equating like-differential forms, shows only $\bar{\lambda}/\lambda$ and $\bar{\kappa}/\kappa$ are linearly independent. The equations are for isotropic media as the same coefficients are used in each differential coordinate direction. The parameters are assumed constant, therefore the medium is homogeneous. Evaluating $dF$ gives

$$\begin{aligned} dF &= \lambda c_m\left(\frac{\partial \zeta_1}{\partial x^2} dx^2 \wedge dx^1 + \frac{\partial \zeta_1}{\partial x^3} dx^3 \wedge dx^1\right) \wedge dx^0 \\ &\quad + \lambda c_m\left(\frac{\partial \zeta_2}{\partial x^3} dx^3 \wedge dx^2 + \frac{\partial \zeta_2}{\partial x^1} dx^1 \wedge dx^2\right) \wedge dx^0 \\ &\quad + \lambda c_m\left(\frac{\partial \zeta_3}{\partial x^1} dx^1 \wedge dx^3 + \frac{\partial \zeta_3}{\partial x^2} dx^2 \wedge dx^3\right) \wedge dx^0 \\ &\quad + \frac{\partial \omega_1}{\partial x^0} dx^0 \wedge dx^2 \wedge dx^3 + \frac{\partial \omega_1}{\partial x^1} dx^1 \wedge dx^2 \wedge dx^3 \\ &\quad + \frac{\partial \omega_2}{\partial x^0} dx^0 \wedge dx^3 \wedge dx^1 + \frac{\partial \omega_2}{\partial x^2} dx^2 \wedge dx^3 \wedge dx^1 \\ &\quad + \frac{\partial \omega_3}{\partial x^0} dx^0 \wedge dx^1 \wedge dx^2 + \frac{\partial \omega_3}{\partial x^3} dx^3 \wedge dx^1 \wedge dx^2 \end{aligned} \tag{B.2}$$

Collecting like terms, we obtain

$$
\begin{aligned}
dF = {} & \lambda c_m \left( \frac{\partial \zeta_3}{\partial x^2} - \frac{\partial \zeta_2}{\partial x^3} \right) dx^1 \wedge dx^2 \wedge dx^0 \\
& + \lambda c_m \left( \frac{\partial \zeta_1}{\partial x^3} - \frac{\partial \zeta_3}{\partial x^1} \right) dx^3 \wedge dx^1 \wedge dx^0 \\
& + \lambda c_m \left( \frac{\partial \zeta_2}{\partial x^1} - \frac{\partial \zeta_1}{\partial x^2} \right) dx^1 \wedge dx^2 \wedge dx^0 \\
& + \frac{\partial \omega_1}{\partial x^0} dx^2 \wedge dx^3 \wedge dx^0 + \frac{\partial \omega_2}{\partial x^0} dx^3 \wedge dx^1 \wedge dx^0 \\
& + \frac{\partial \omega_3}{\partial x^0} dx^1 \wedge dx^2 \wedge dx^0 \\
& + \left( \frac{\partial \omega_1}{\partial x^1} + \frac{\partial \omega_2}{\partial x^2} + \frac{\partial \omega_3}{\partial x^3} \right) dx^1 \wedge dx^2 \wedge dx^3.
\end{aligned}
\tag{B.3}
$$

By using the Hodge-$\star$ to obtain a 4-vector representation of these equations, one obtains from $dF = 0$:

$$\nabla \cdot \omega = 0, \; \nabla \times \zeta + \frac{\partial \omega}{\lambda c_m \partial t} = 0. \tag{B.4}$$

We notice $\left[ \lambda c_m \right] = 1$. Similarly, by taking the Hodge-$\star$ of the first of equation B.1), using the facts $^{\star\star}H = -H$ and $^{\star\star}J = J$, then proceeding to evaluate $dH = 4\pi J$, one obtains

$$\nabla \cdot \bar{\zeta} = \frac{4\pi}{\bar{\lambda}\bar{\eta}} c_m \rho, \quad -\frac{1}{c_m} \frac{\partial \bar{\zeta}}{\partial t} + \nabla \times \frac{\bar{\kappa}}{\bar{\lambda}} \bar{\omega} = \frac{4\pi}{\bar{\lambda}\bar{\eta}} J. \tag{B.5}$$

We use dimensional analysis to verify a consistent system of units can be obtained in which $\bar{\eta}$ can be eliminated. With the basic units of mass ($M$), length ($L$) and time ($T$) we expect to find at most three independent parameters. In the main part, the units are chosen so that $\omega$ and $\zeta$ have the same units as the NST quantities $\Omega$ and $\Lambda$: $[\omega] = T^{-1}$, $[\zeta] = LT^{-2}$. Furthermore, $[J] = M/L^2T$, and $[\lambda] = TL^{-1}$, $[\kappa] = 1$. We also have $[\eta] = [M/LT] = ET/L^3$ and $[\bar{\eta}] = [M/L] = [\eta]T$, and $[A] = L/T = [u]$. Thus $\left[ \lambda c_m \right] = 1$. Equation (B.4) is thus dimensionally consistent. The fluid dynamical Lorentz force (density) law is then found:

$$f^i = \lambda J_0 \zeta^i + \kappa (J \times \omega)^i = J_0 \left( \lambda \zeta^i + \frac{\kappa}{c_m} (u \times \omega)^i \right). \tag{B.6}$$

Where $J_0 = \rho c_m$ and in the second equation we use $J^i = \rho u^i$, a relation valid for $\| u \| \ll c_m$, requires the sum term be dimensionally homogeneous, so as a check we have

$$[\lambda \zeta] \oplus \left[ \frac{\kappa}{c_m} u \times \omega \right] \Rightarrow \frac{T}{L} \frac{L}{T^2} \oplus \frac{T}{L} \frac{L}{T} \frac{1}{T} = \frac{1}{T}. \tag{B.7}$$

To maintain this relationship, we cannot set $[\lambda] = 1$. From the first of equation B.5), we see a simplification $\bar{\lambda}\bar{\eta} = c_m$ is possible. So $\left[ \bar{\lambda} \right] = L^2 M^{-1} T^{-1}$. This requires $L^{-1} \left[ \bar{\zeta} \right] = ML^{-3}$, so $\left[ \bar{\zeta} \right] = ML^{-2}$. Using $\bar{\lambda}\bar{\eta} = c_m$, in the second of equation (B.5) we find

$$\frac{T}{L} \cdot \frac{M}{L^2} \cdot \frac{1}{T} \oplus \left[ \nabla \times \frac{\bar{\kappa}}{\bar{\lambda}} \bar{\omega} \right] = \frac{M}{L^3}. \tag{B.8}$$

This relation is consistent as long as $\left[ \nabla \times \frac{\bar{\kappa}}{\bar{\lambda}} \bar{\omega} \right] = ML^{-3}$. So that defining $\varpi \equiv \left[ \bar{\kappa}/\bar{\lambda} \right] \bar{\omega} = ML^{-2}$ and using $c_m^2 = \kappa \bar{\lambda} / \bar{\kappa} \lambda$ from corollary 3 and $\kappa = 1$, as well as

$[\lambda] = TL^{-1}$, we see $\left[\bar{\kappa}/\bar{\lambda}\right] = TL^{-1}$ and $\bar{\kappa} = LM^{-1}$. Summarizing:

$$\left[\frac{\varpi}{L}\right] = \frac{M}{L^3}, \left[\frac{\bar{\zeta}}{L}\right] = \frac{M}{L^3}. \tag{B.9}$$

These results suggest the vortex field excitations are ones of mass density. The fields themselves are mass fluxes defined as mass through an area. Since $\nabla \times \varpi$ and $\nabla \times \bar{\zeta}$ have these units, it may well be that these curl variables are more closely related to physically measurable variables. One can verify $[\lambda\zeta] = \left[\bar{\lambda}\bar{\zeta}\right] = T^{-1}$ and $[\kappa\omega] = \left[\bar{\kappa}\bar{\omega}\right] = T^{-1}$, so the constitutive relations are consistent with the present reduction. The field equations can then be written in the Maxwell equation form by using the new variable $\varpi = \bar{\lambda}^{-1}\omega$ introduced above:

$$\begin{array}{ll} \nabla \cdot \omega = 0 & \nabla \times \zeta + \dfrac{\partial \omega}{\partial t} = 0 \\ \nabla \cdot \bar{\zeta} = 4\pi\rho & -\dfrac{1}{c_m}\dfrac{\partial \bar{\zeta}}{\partial t} + \nabla \times \varpi = \dfrac{4\pi}{c_m} J \end{array} \tag{B.10}$$

Minimal Parameter Set VFEs

Thus, by choosing the scales of the fields, the fluid VFEs map to the form of the electromagnetic VFEs. The three parameters $\lambda\bar{\lambda}^{-1}$, $\bar{\lambda}^{-1}$, and $\bar{\kappa}$ allow $c_m$ to be computed and enter the theory as follows:

$$\bar{\zeta} = \left(\lambda\bar{\lambda}^{-1}\right)\zeta,\ \varpi = \left(\bar{\kappa}/\bar{\lambda}\right)\bar{\omega} = \bar{\lambda}^{-1}\omega,\ c_m^2 = \kappa\bar{\lambda}/\bar{\kappa}\lambda = 1/\left(\bar{\kappa}\lambda\bar{\lambda}^{-1}\right). \tag{B.11}$$

With our restriction of the units for the vortex field to match the NST variables, the units of the excitations are thereby determined by the dynamical equations. The choice of metric $(g_{\mu\nu}) = (1, -1, -1, -1)$ used here is reflected in the signs of the density $\rho$ (+) and current $J$ (+). For time-dependent flows with many spatial scales, it is likely that the constitutive parameters $\left\{\bar{\kappa}/\bar{\lambda}, \lambda\right\}$ and $c_m^2 = \kappa\bar{\lambda}/\bar{\kappa}\lambda$ are frequency and wave vector dependent as are the analogous parameters in electrodynamics.

## References


[1] Scofield D F and Huq P 2014 Fluid dynamical Lorentz force law and Poynting theorem—introduction *Fluid Dyn. Res.* **47** 055513

[2] Landau L D and Lifshitz E M 1989 *Fluid Mechanics* 2nd ed (with corrections) (Oxford: Butterworth-Heineman)

[3] Scofield D F and Huq P 2010 Concordance among electromagnetic, fluid dynamical, and gravitational theories *Phys. Lett.* A **374** 3476–82

[4] Jackson J D 1998 *Classical Electrodynamics* 3rd ed (New York: Wiley)

[5] Marmanis H 1998 Analogy between the Navier–Stokes equations and Maxwell's equations: application to turbulence *Phys. of Fluids* **10** 1428–37
Errata 1998 *Phys. of Fluids* **10** 3031

[6] Belevich M 2008 Non-relativistic abstract continuum mechanics and its possible physical interpretations *J. Phys. A: Math. Theor.* **41** 045401

[7] Martins A A and Pinheiro M J 2009 Fluidic electrodynamics: an approach to electromagnetic propulsion *Phys. of Fluids* **21** 097103

[8] Kambe T 2010 A new formulation of equations of compressible fluids by analogy with Maxwell's equations *Fluid Dyn. Res.* **42** 055502

[9] Kambe T 2010 *Geometrical Theory of Dynamical Systems and Fluid Flows* (Singapore: World Scientific)

[10] Eckart C 1940 The thermodynamics of irreversible processes. Part III. Relativistic theory of simple fluids *Phys. Rev.* **58** 919–24

[11] Carter B 1973 Elastic perturbation theory in general relativity and a variational principle for a rotating solid star *Commun. Math. Phys.* **30** 261–86
[12] Stewart J M 1977 On transient relativistic thermodynamics and kinetic theory *Proc. Roy. Soc. (London)* A **347** 59–75
[13] Israel W and Stewart J M 1977 On transient relativistic thermodynamics and kinetic theory II *Proc. Roy. Soc (London)* A **365** 43–52
[14] Hiscock W A and Lindblom L 1985 Generic instabilities in first-order dissipative relativistic fluid theories *Phys Rev.* D **31** 725–33
[15] Liu I-S, Muller I and Ruggeri T 1986 Relativistic thermodynamics of gases *Ann. Phys.* **169** 191–219
[16] Carter B 1989 *Relativistic Fluid Dynamics* ed A Anile and Y Choquet-Bruhat (Berlin: Springer)
[17] Geroch R and Lindblom L 1990 Dissipative relativistic fluid theories of divergence type *Phys. Rev.* D **41** 1855–61
[18] Lindblom L 1996 The relaxation effect in dissipative relativistic fluid theories *Ann. Phys.* **247** 1–18
[19] Lindsey J E, Wands D and Copeland E J 2000 Superstring cosmology *Phys. Rep.* **331** 343–492
[20] Baranskii K N, Sever G A and Velichkina T S 1971 Propagation of transverse hypersonic waves in low-viscosity liquids *JETF Pis. Red.* **13** 52–54
[21] Berdyev A A and Lezhnev I B 1971 Transverse sound in liquids *JETF Pis. Red.* **13** 49–51
[22] Hosokawa S *et al* 2009 Transverse acoustic excitations in Liquid Ga *Phys. Rev. Lett.* **102** 105502
[23] Hehl F W and Obukhov Y N 2003 *Foundations of Classical Electrodynamics* (Boston: Birkhauser)
[24] Moffatt H K 1969 The degree of knottedness of tangled vortex lines *J. Fluid Mech.* **35** 117–29
[25] Moffatt H K and Tsinober A 1992 Helicity in laminar and turbulent flow *Ann. Rev. Fluid Mech.* **24** 281–312
[26] Levich E and Tsinober A 1983 On the role of helical structures in three-dimensional turbulent flow *Phys. Lett.* A **93** 293–7
[27] Branover H, Eidelman A, Golbraikh E and Moiseev S 1999 *Turbulence and Structures, Chaos, Fluctuations, and Helical Self-Organization in Nature and the Laboratory* (New York: Academic Press)
[28] Scofield D F and Huq P 2010 Evolution of helicity in fluid flows *J. Math. Phys.* **51** 033520
[29] Kiehn R M 2007 *Non-equlibrium Thermodynamics* (Morrisville, N.C.: Lulu Enterprises,)
[30] Ricca R L (ed) 2001 An introduction to the geometry and topology of fluid flows *NATO ASI Series II (Cambridge, UK)* vol 47 (The Netherlands: Kluwer, Dordrect)
[31] Lectures on Topological fluid dynamics Ricca R L (ed) 2009 *Springer CIME Lecture Notes on Mathematics* (Heidelberg: Springer-Verlag)
[32] Landau L D and Lifshitz E M 2004 *The Classical Theory of Fields* 4th ed (Amsterdam: Elsevier)
[33] Misner C W, Thorne K S and Wheeler J A 1973 *Gravitation* (San Francisco: W H Freeman)
[34] Montesinos M and Flores E 2006 Symmetric energy-momentum tensor in Maxwell, Yang-Mills, and Proca theories obtained using only Noether's theorem, Revista Mexicana de Fisica **52** 29–36 (http://arxiv.org/abs/hep-th/0602190).
[35] Yudovich V I 2000 On the loss of smoothness of the solutions of the Euler equations and the inherent instability of flows of an ideal fluid *Chaos* **10** 705–15
[36] Tsinober A 2012 *An Informal Introduction to Turbulence* 2nd ed (Dordrecht: Kluwer)
[37] Scofield D F and Huq P 2009 Transverse waves and vortex fields in non-relativistic fluid flows *Phys. Lett.* A **373** 1155–8
[38] Flanders H 1989 *Differential Forms with Applications to the Physical Sciences* (New York, NY: Dover)
[39] Berger S A, Talbot L and Yao L S 1983 Flow in curved tubes *Ann. Rev. Fluid Mech.* **15** 461–512